J. N. M.

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EDWARD D. MANSFIELD.

No

THE

ELEMENTS

OF A

NEW ARITHMETICAL NOTATION,

AND OF A

NEW ARITHMETIC OF INFINITES:

In Two Books:

IN WHICH THE SERIES DISCOVERED BY MODERN MATHEMATICIANS, FOR THE QUADRATURE OF THE CIRCLE AND HYPERBOLA, ARE DEMONSTRATED TO BE AGGREGATELY INCOMMENSURABLE QUANTITIES: AND A CRITERION IS GIVEN, BY WHICH THE COMMENSURABILITY OR INCOMMENSURABILITY OF INFINITE SERIES MAY BE ACCURATELY ASCERTAINED.

WITH

AN APPENDIX,

CONCERNING SOME PROPERTIES OF PERFECT, AMICABLE, AND OTHER NUMBERS, NO LESS REMARKABLE THAN NOVEL.

BY THOMAS TAYLOR.

Ὁ βιος ανθρωποις λογισμου κ' αριθμου δειται πανυ·
Ζωμεν δ' αριθμῳ και λογισμῳ, ταυτα γαρ σωζει βροτους.

EPICHARMUS.

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PREFACE.

If novelty alone were sufficient to entitle a work to applause, of whatever description it may be, the following pages would not, perhaps, rank among those productions that merit only a small degree of praise; but as novelty in mathematics, unaccompanied by truth, is perfectly nugatory, and resembles "a tale told by an idiot, signifying nothing," the author of this elementary treatise trusts that his theory will be found to be no less true than novel, and no less rigidly accurate in its demonstrations than important in its results.

The new mode of notation adopted in this work, depends on numeral interval, or vacuity in connexion with number, and position with reference to unity. Thus, let there be given an

infinite series of units, $1+1+1+1+1+1$, &c., and under this another series, with an interval between each of the units as follows, $.\,1+.\,1+.\,1+.\,1+.\,1$, &c.; then it is evident that the latter series will be one half of the former, *i. e.* it will be equal to $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$, &c. Hence $.\,1$ will be equivalent to $\frac{1}{2}$. As likewise $..\,1+..\,1+..\,1+..\,1+..\,1$, &c., is one third of the series $1+1+1+1+1+1$, &c., it follows that $..\,1$ is equivalent to $\frac{1}{3}$. And in a similar manner $...\,1$ will be equivalent to $\frac{1}{4}$, $....\,1$ to $\frac{1}{5}$, and so on *ad infinitum*. Hence, as $.\,1+..\,1+...\,1+....\,1$, &c. is equivalent to $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, &c., if this series is considered with reference to unity at the beginning of it, viz. if, conformably to decimal notation, we place 0 before the first term of the series, as indicative of the place of unity, then the series will be $0+.\,1+..\,1+...\,1+....\,1$, &c., and will, by referring the several terms to unity, be equivalent to the series $\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}$, &c. For the first unit is in the third

place from unity, the second is in the sixth, the third in the tenth place; and so of the rest *.

Two of the results of this mode of notation, and which I conceive to be the most important, are these: that the series discovered by modern mathematicians, for the quadrature of the circle and hyperbola, are shown by it to be aggregately incommensurable quantities; and that it affords a criterion by which the commensurability or in-

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* As 1 . + 1 . + 1 . + 1 ., &c. is also one half, 1 . . + 1 . . + 1 . . + 1 . ., &c. is one third, 1 . . . + 1 . . . + 1 . . . + 1 . . ., &c. is one fourth of the series 1+1+1+1+1, &c., I might have used this notation instead of the above; and in this case the series 1 . + 1 . . + 1 . . . + 1 . . . ., &c. would be equivalent to the series 1 + . 1 + . . 1 + . . . 1 + ... 1, &c., by considering the second unit as in the third place from the first unit, and as therefore indicating $\frac{1}{3}$; the third unit as in the sixth place from the first unit, and designating $\frac{1}{6}$; and so of the rest of the terms. I have, however, adopted the former notation as more analogous to that of decimals.

commensurability of infinite series may be infallibly and universally ascertained.

In what is said in the Second Book respecting the difference between the *distributed* and *undistributed* value of infinite series, it must be observed, as I have there remarked, that though the propositions, in Dr. Wallis's Arithmetic of Infinites, are not true of the series which he adduces, according to the *former*, yet they are according to the *latter* value, *i. e.* according to an aggregation of the terms into one sum. What I have said, therefore, does not at all invalidate his reasoning by induction, or subvert the conclusions which that reasoning produces.

As the present work professes to be nothing more than an outline of a New Arithmetical Notation, other conclusions will, I trust, be found to result from it, no less remarkable and important than those contained in the following pages. But the accomplishment of this I shall

leave to those who may think my discovery worthy of their attention, and who are professedly mathematicians. For, as the promulgation of the philosophy of Plato and Aristotle has been, for the far greater part of my life, and is still, the mark at which all my efforts ultimately aim, I can only say on this subject, that it is sufficient for me to have opened the fountains, and to leave the ramifications of them to others.

ERRATUM.

The reader is requested to correct the series in p. 15, line 4 from the bottom, as follows: $1 + \frac{3}{6} + \frac{11}{36} + \frac{50}{240} + \frac{50}{360} + \frac{50}{504}$, &c.

THE ELEMENTS

OF A

NEW ARITHMETICAL NOTATION,

ETC. ETC.

BOOK THE FIRST.

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1. If the infinite series $1+1+1+1+1+1$, &c. is divided by $1+1$, by $1+1+1$, by $1+1+1+1$, and so on, *ad infinitum*, the quotients $1\,.$, $1\,..$, $1\,...$, &c. will be equal to $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c.

For $\frac{1+1+1+1+1, \&c.}{1+1} = 1\,.+1\,.+1\,.+1\,.$, &c.

But $\frac{1+1+1+1+1, \&c.}{1+1} = \frac{1}{1+1} + \frac{1}{1+1} + \frac{1}{1+1} + \frac{1}{1+1}$, &c. *ad infin.* And therefore $1\,.$ is equal to $\frac{1}{1+1}$.

Thus also $\frac{1+1+1+1+1+1, \&c.}{1+1+1} = 1\,..+1\,..+1\,..+1\,..$ &c. And $\frac{1+1+1+1+1+1, \&c.}{1+1+1} =$

$\frac{1}{1+1+1}+\frac{1}{1+1+1}+\frac{1}{1+1+1}$, &c. Hence 1 . . is equal to $\frac{1}{1+1+1}$. And so of the rest.

2. Not only, however, points placed after unity cause unity to be of a fractional value, but this is also true of points placed before unity.

For as 1 . + 1 . + 1 . + 1 ., &c. is equal to an infinite series of $\frac{1}{2}$, so likewise is .1 + .1 + .1 + .1 + .1, &c.; because the latter, as well as the former series, is evidently one-half of the series 1+1+1+1+1+1, &c.; and consequently . 1 is equal to $\frac{1}{2}$.

The only difference between 1.+1.+1.+1., &c. and .1 + .1 + .1 + .1 + .1, &c. is this, that the former series terminates in a point, and the latter begins with it. Hence the latter is nothing more than the former series inverted.

3. Hence the infinite series 1+.1+..1+...1+....1, &c. *i. e.* $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c. will, by a summation of the terms, become 1+1+1+1+1+1+1, &c. viz.:—

```
1
.+1
.  .+1
.  .  .+1
.  .  .  .+1
.  .  .  .  .+1, &c.
----------------------
= 1+1+1+1+1+1, &c.
```

And thus the infinite series $1+1+1+1+1+1$, &c. will either indicate an infinite series of units, or the infinite fractional series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c.; just as the infinite series 1 1 1 1 1 1 1 1 1, &c. either designates whole numbers, or the decimal of $\frac{1}{9}$. Thus, too, the infinite series $.1+ \ldots 1+ \ldots\ldots 1+ \ldots\ldots\ldots 1$, &c. $=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}$, &c. will, by a summation of the terms, become the series $.+1\,.+1\,.+1\,.+1$, &c. viz.: —

$$\begin{array}{lllll}
 & .+1 & & & \\
 & . \quad . & .+1 & & \\
 & . \quad . \quad . & . \quad . & .+1 & \\
 & . \quad . \quad . \quad . & . \quad . & . \quad . & .+1 \\
\hline
= & .+1 & .+1 & .+1 & .+1, \text{\&c.}
\end{array}$$

And the infinite series $1+\ldots 1+\ldots\ldots 1+\ldots\ldots\ldots 1$, &c. $=1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}$, &c. will become, by summation, the series $1\,.+1\,.+1\,.+1$, &c. viz.: —

$$\begin{array}{lllll}
 & 1 & & & \\
 & . & .+1 & & \\
 & . & . \quad . & .+1 & \\
 & . & . \quad . & . \quad . & .+1, \text{\&c.} \\
\hline
= & 1 & .+1 & .+1 & .+1, \text{\&c.}
\end{array}$$

Hence the series $.+1\,.+1\,.+1\,.+1$, &c.; or, which is the same thing, the series $.1+\,.1+\,.1+\,.1+$, &c, will, according to a fractional value, be equal to $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}$, &c. For the first term .1, is evidently $\frac{1}{2}$; the second term .1, is, from

position with reference to unity, $\frac{1}{4}$; for it is by position equivalent to . . . 1. Thus, too, the third term will be equal to $\frac{1}{6}$, the fourth to $\frac{1}{8}$, and so on. In like manner the series 1 . + 1 . + 1 . + 1 . + 1, &c.; or 1 + . 1 + . 1 + . 1 + . 1, &c. which is equivalent to it, will, by considering the position of the terms with reference to unity, be equal to $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}$, &c. Again, 1 . . + 1 . . + 1 . . + 1 . . + 1, &c.; or, which is equivalent, 1 + . . 1 + . . 1 + . . 1 + . . 1, &c. will be equal to the infinite series $1+\frac{1}{4}+\frac{1}{7}+\frac{1}{10}+\frac{1}{13}$, &c. And 1 + . . . 1 + . . . 1 + . . . 1 + . . . 1, &c. will be $1+\frac{1}{5}+\frac{1}{9}+\frac{1}{13}+\frac{1}{17}$, &c. Hence an infinite series of units, with one interval between each, will be either the series $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}$, &c. or the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}$, &c. A series with two intervals between each unit, will either be the series $1+\frac{1}{4}+\frac{1}{7}+\frac{1}{10}+\frac{1}{13}$, &c. or the series $\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}$, &c. if the first term of the latter series is . . 1, *i. e.* $\frac{1}{3}$, instead of 1; so as to be the series . . 1 + . . 1 + . . 1 + . . 1, &c. A series with three intervals, will either be the series $1+\frac{1}{5}+\frac{1}{9}+\frac{1}{13}+\frac{1}{17}$, &c. or the series $\frac{1}{4}+\frac{1}{8}+\frac{1}{12}+\frac{1}{16}+\frac{1}{20}$, &c. And so in other instances, *ad infinitum*.

4. The expressions from which these series are evolved are the following: $\frac{1-.1}{1-1}$; which, when evolved, is $1+.1+.1+.1+.1$, &c. $=1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}$, &c., the numerator of the expression being equivalent to $1-\frac{1}{2}$, or $\frac{1}{2}$. Another expression is $\frac{.1}{1-1}$; which, when evolved, is the series $.1+.1+.1+.1+.1$, &c. $=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}$, &c. And hence, since the numerator $.1$ is equal to $1-.1$, *i. e.* is equal to $\frac{1}{2}$; what modern mathematicians have demonstrated of these two series, viz. that they are equal to each other, is immediately evident. Again, $\frac{1-..2}{1-1}$ is, when expanded, the series $1+..1+..1+..1+..1$, &c. $=1+\frac{1}{4}+\frac{1}{7}+\frac{1}{10}+\frac{1}{13}$, &c. And $\frac{..1}{1-1}$ is $..1+..1+..1+..1+..1$, &c. $=\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}+\frac{1}{18}$, &c. And the former series is equal to the latter; for $1-..2$, *i. e.* $1-\frac{2}{3}=\frac{1}{3}=..1$. Thus, too, $\frac{1-...3}{1-1}$ is, when expanded, $1+...1+...1+...1+...1+...1$, &c. $=1+\frac{1}{5}+\frac{1}{9}+\frac{1}{13}+\frac{1}{17}+\frac{1}{21}$, &c. And $\frac{...1}{1-1}$ will be, when evolved, $...1+...1+...1+...1+...1$, &c. $=\frac{1}{4}+\frac{1}{8}+\frac{1}{12}+\frac{1}{16}+\frac{1}{20}$, &c.; the latter being equal to the former series. For $1-...3$, *i. e.* $1-\frac{3}{4}=\frac{1}{4}=...1$.

Thus also $\frac{1-\ldots.4}{1-1}$ is, when evolved, $1+\ldots.1+\ldots.1+\ldots.1+\ldots.1$, &c. $=1+\frac{1}{6}+\frac{1}{11}+\frac{1}{16}+\frac{1}{21}+\frac{1}{26}$, &c. And $\frac{\ldots.1}{1-1}$ is $\ldots.1+\ldots.1+\ldots.1+\ldots.1$, &c. $=\frac{1}{5}+\frac{1}{10}+\frac{1}{15}+\frac{1}{20}+\frac{1}{25}$, &c. And the latter is equal to the former series; because $1-\ldots.4=1-\frac{4}{5}=\frac{1}{5}=\ldots.1$. In like manner it may be shown by this method, that in all series, which are parts of the infinite series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c. the value of which is infinite, any two of such series, whose denominators are in an arithmetical progression, and of which those in one series differ from those in the other by an equal difference, are equal to each other; a property which has not before been demonstrated of this series.

5. It may also be shown, independently of this method, and in a way which at the same time confirms it, that such series are equal to each other. For all such series being parts of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c., and the value of this series being infinite, the value of those series will likewise be infinite. That the series therefore $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}$, &c., is equal to the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}$, &c., an infinitesimal excepted,

may be demonstrated by subtracting the latter from the former series, as below:—

$$\begin{array}{l} 1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}, \text{ \&c.} \\ \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}, \text{ \&c.} \\ \hline \frac{1}{2}+\frac{1}{12}+\frac{1}{30}+\frac{1}{56}+\frac{1}{90}+\frac{1}{132}, \text{ \&c.} \end{array}$$

Which remainder is the half of the series $1+\frac{1}{6}+\frac{1}{15}+\frac{1}{28}+\frac{1}{45}+\frac{1}{66}$, &c.; and this series is a part of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\frac{1}{28}+\frac{1}{36}+\frac{1}{45}+\frac{1}{55}+\frac{1}{66}$, &c. But the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$, &c., is, as is well known, equal to 2; and, consequently, the series $\frac{1}{2}+\frac{1}{12}+\frac{1}{30}+\frac{1}{56}$, &c. is less than 1; and is therefore but an infinitely small part of a series whose value is infinite.

Thus, too, if from the series $1+\frac{1}{4}+\frac{1}{7}+\frac{1}{10}+\frac{1}{13}+\frac{1}{16}$, &c. the series $\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}+\frac{1}{18}$, &c. is subtracted, the remainder will be as below:—

$$\begin{array}{l} 1+\frac{1}{4}+\frac{1}{7}+\frac{1}{10}+\frac{1}{13}+\frac{1}{16}, \text{ \&c.} \\ \quad\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}, \text{ \&c.} \\ \hline 1-\frac{1}{12}-\frac{1}{42}-\frac{1}{90}-\frac{1}{156}-\frac{1}{240}, \text{ \&c.} \end{array}$$

But the double of the terms $\frac{1}{12}$, $\frac{1}{42}$, $\frac{1}{90}$, $\frac{1}{156}$, $\frac{1}{240}$, &c. will be the terms $\frac{1}{6}$, $\frac{1}{21}$, $\frac{1}{45}$, $\frac{1}{78}$, $\frac{1}{120}$, &c. which form a part of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}$, &c., and will therefore be less than $\frac{1}{2}$. Again, if from the series $1+\frac{1}{5}+\frac{1}{9}+\frac{1}{13}+\frac{1}{17}+\frac{1}{21}$, &c. the series $\frac{1}{4}+\frac{1}{8}+\frac{1}{12}+\frac{1}{16}+\frac{1}{20}$, &c. is subtracted, the remainder will be $1-\frac{1}{20}-\frac{1}{72}-\frac{1}{156}-\frac{1}{272}-\frac{1}{420}$, &c.; and the double of these frac-

tional terms will be the terms $\frac{1}{10}$, $\frac{1}{36}$, $\frac{1}{78}$, $\frac{1}{136}$, $\frac{1}{210}$, &c., which also form a part of the series $1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21}$, &c. and will be less than $\frac{1}{2}$, by a deficiency greater than that of the preceding remainder. Thus, too, if from the series $1 + \frac{1}{6} + \frac{1}{11} + \frac{1}{16} + \frac{1}{21} + \frac{1}{26}$, &c. the series $\frac{1}{5} + \frac{1}{10} + \frac{1}{15} + \frac{1}{20} + \frac{1}{25}$, &c. is subtracted, the remainder will be $1 - \frac{1}{30} - \frac{1}{110} - \frac{1}{240} - \frac{1}{420} - \frac{1}{650}$, &c.; and the double of these fractional terms will be the terms $\frac{1}{15}$, $\frac{1}{55}$, $\frac{1}{120}$, $\frac{1}{210}$, $\frac{1}{325}$, &c., which form a part of the series $1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21}$, &c., and will be in a still greater degree less than $\frac{1}{2}$ than the preceding remainder. And the like will be found to take place in all the other parts of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$, &c. *ad infinitum.* By this method of notation, therefore, it may be easily shown, that in the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$, &c., any series whose denominators differ from each other by 2, 3, 4, 5, 6, &c. are the $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, &c. parts of that series. And it may also be proved, independently of this method, by subtracting one series from the other, in the same way as above. Thus it may be easily demonstrated, that the series $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \frac{1}{15} + \frac{1}{18}$, &c., the series $1 + \frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \frac{1}{13} + \frac{1}{16}$, &c.,

and the series $\frac{1}{2}+\frac{1}{5}+\frac{1}{8}+\frac{1}{11}+\frac{1}{14}+\frac{1}{17}$, &c., are equal to each other; and that each is $\frac{1}{3}$ of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c., because an infinitely small quantity only will remain in subtracting them from each other. Thus, too, it will be found that the series $1+\frac{1}{5}+\frac{1}{9}+\frac{1}{13}+\frac{1}{17}+\frac{1}{21}$, &c., the series $\frac{1}{2}+\frac{1}{6}+\frac{1}{10}+\frac{1}{14}+\frac{1}{18}+\frac{1}{22}$, &c., the series $\frac{1}{3}+\frac{1}{7}+\frac{1}{11}+\frac{1}{15}+\frac{1}{19}+\frac{1}{23}$, &c., and the series $\frac{1}{4}+\frac{1}{8}+\frac{1}{12}+\frac{1}{16}+\frac{1}{20}+\frac{1}{24}$, &c., are equal to each other, an infinitesimal excepted; and also that each is equal to $\frac{1}{4}$ of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c. And so in all other instances.

6. Points placed before the affirmative or negative signs in the denominators of expressions, which, when evolved, produce infinite series, the value of which is infinite, do not cause the terms of such denominators to have a fractional value, but cause them to be the double, triple, quadruple, and, in short, to be multiples of what they were before. Thus $1\,.-1$ is the double of $1-1$: for $\frac{1\,.-1}{1-1}=1+1$, $1\,..-1$ is the triple of $1-1$, $1\,...-1$ is the quadruple of $1-1$, and so on. Thus, too, $1\,.-2\,.+1$ is quadruple of $1-2+1$: for it is produced by the multiplication

of $1 . -1$ into itself, and when divided by $1-2+1$, gives for the quotient $1+2+1$. And $1 . . -2 . . +1$ is nine times $1-2+1$. But when the numerators of such expressions consist of more terms than one, then points placed before the affirmative or negative signs of them do not alter their value, an infinitesimal excepted, as will be shortly evident.

7. What has been above inferred of the infinite series, which are parts of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c., may also be inferred of the infinite series, which are parts of the infinite series $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\frac{6}{7}+\frac{7}{8}+\frac{8}{9}$, &c. For the series $\frac{2}{3}+\frac{4}{5}+\frac{6}{7}+\frac{8}{9}$, &c., minus the series $\frac{1}{2}+\frac{3}{4}+\frac{5}{6}+\frac{7}{8}$, &c., leaves, as the remainder, the series $\frac{1}{6}+\frac{1}{20}+\frac{1}{42}+\frac{1}{72}$, &c., and this, multiplied by 2, is equal to $\frac{1}{3}+\frac{1}{10}+\frac{1}{21}+\frac{1}{36}$, &c., a part of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{4}{21}$, &c., and, consequently, the above two series are equal to each other, an infinitesimal excepted. Thus, too, $\frac{2}{3}+\frac{5}{6}+\frac{8}{9}+\frac{11}{12}+\frac{14}{15}$, &c., minus $\frac{1}{2}+\frac{4}{5}+\frac{7}{8}+\frac{10}{11}+\frac{13}{14}$, &c., leaves $\frac{1}{6}+\frac{1}{30}+\frac{1}{72}+\frac{1}{132}+\frac{1}{210}$, &c.; and this, multiplied by 2, produces the series $\frac{1}{3}+\frac{1}{15}+\frac{1}{36}+\frac{1}{66}+\frac{1}{105}$, &c.; and therefore these two series are equal to each other, an infinitely small part excepted. Hence, in the

series $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}$, &c., in the same manner as in the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, &c., any infinite series of terms, whose denominators differ from each other by 2, 3, 4, 5, &c., are the half, third, fourth, &c. parts of that series. But the series $\frac{1}{2}+\frac{3}{4}+\frac{5}{6}+\frac{7}{8}$, &c., is produced by the expansion of the expression $\frac{0+1\,.+1}{1\,.-2\,.+1}=0+1\,.+3\,.+5\,.+7$, &c. And the series $\frac{2}{3}+\frac{4}{5}+\frac{6}{7}+\frac{8}{9}$, &c. is produced by the expansion of the expression $\frac{0+0+2}{1\,.-2\,.+1}=0+0+2\,.+4\,.+6\,.+8\,.$, &c. Hence the expression $\frac{0+0+2}{1\,.-2\,.+1}$ is equal to the expression $\frac{0+1\,.+1}{1\,.-2\,.+1}$. But the series $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}$, &c., is produced by the expansion of the expression $\frac{0+1}{1-2+1}=0+1+2+3+4+5+6$, &c., which is double of the expression $\frac{0+1\,.+1}{1\,.-2\,.+1}$, and of $\frac{0+0+2}{1\,.-2\,.+1}$. For the numerator $0+1$ is the half of each of the numerators $0+1\,.+1$ and $0+0+2$, and the denominator $1-2+1$ is $\frac{1}{4}$ of the denominator $1\,.-2\,.+1$. The expression also, from which the series $\frac{2}{3}+\frac{5}{6}+\frac{8}{9}+\frac{11}{12}+\frac{14}{15}$, &c. is expanded, is $\frac{0+0+2\,..+1}{1\,..-2\,..+1}=0+0+2\,..+5\,..+8\,..+11\,..+14\,..$, &c. And the expression which, when

expanded, produces the series $\frac{1}{2}+\frac{4}{5}+\frac{7}{8}+\frac{10}{11}+\frac{13}{14}$, &c. is $\frac{0+1\,..+2}{1\,..-2\,..+1}$; both which expressions are equal to each other. But each of the expressions $\frac{0+0+2\,..+1}{1\,..-2\,..+1}$, $\frac{0+1\,..+2}{1\,..-2\,..+1}$, is one third of the expression $\frac{0+1}{1-2+1}$. For the numerators $0+0+2\,..+1$, $0+1\,..+2$, are each of them triple of the numerator $0+1$; and the denominator $1\,..-2\,..+1$ is nine times the denominator $1-2+1$: for the former, divided by the latter, gives $1+2+3+2+1$. And each of the expressions $\frac{0+1\,.+1}{1\,.-2\,.+1}$, $\frac{0+0+2}{1\,.-2\,.+1}$, is one half of the expression $\frac{0+1}{1-2+1}$. For each of the numerators $0+1\,.+1$, $0+0+2$, is double of the numerator $0+1$, and the denominator $1\,.-2\,.+1$ is quadruple the denominator $1-2+1$. Hence the two former of these equal series are each of them equal to one half of the series $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}$, &c., and the two latter are each of them equal to one third of that series. Both the numerators also, and denominators of the two former equal series, differ from each other by 2, and those of the latter by 3.

8. According to this method of notation $\frac{1}{1-\,.1}$,

when expanded, will be $1+.1+..1+...1+....1$, &c. $=1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}$, &c. But the expression $\frac{1}{1-.1}$ is equivalent to $\frac{1}{1-\frac{1}{2}}=2$; and, consequently, the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$, &c. is equal to 2; which modern mathematicians have demonstrated to be true of this series. And from this mode of notation also it follows, that the multiplication, in producing the terms of an infinite series which is a quotient, is analogous to the multiplication of letters in algebra. Thus, in the division of 1 by $1-.1$, which produces the above quotient, the process is as below:—

```
1—.1) 1      (1+.1+..1+...1+....1, &c.
      1—.1
      ————
       +.1
       +.1—..1
       ———————
           +..1
           +..1—...1
           —————————
                +...1
                +...1—....1
                ———————————
                      +....1, &c.
```

Here it is evident that the second term $+.1$ of the quotient, multiplied by $-.1$ of the divisor, produces $-..1$, viz. $+\frac{1}{2}\times-\frac{1}{2}$ produces $-\frac{1}{3}$, just as $+1a\times-1a$ produces $-1aa$. The third term of the quotient, viz. $+..1\times-.1$ of the divisor produces $-...1$, just as $+1aa\times-1a$

produces $-1aaa$. And so in all other instances. But then, from the position of the terms of the quotient with reference to unity, . 1 or $\frac{1}{2}$ becomes . . 1, or $\frac{1}{3}$, . . 1 becomes . . . . . 1 or $\frac{1}{6}$, and so of the rest. The following instances, likewise, are confirmations of the truth of this method of notation : —

$$
\begin{array}{l}
1-.1)\ 1+.1+..1\ (1+.2+..3+...3+....3+.....3,\ \&c.\\
\quad\quad\ \underline{1-.1}\\
\quad\quad\quad +.2+..1\\
\quad\quad\quad \underline{+.2-..2}\\
\quad\quad\quad\quad\quad +..3\\
\quad\quad\quad\quad\quad \underline{+..3-...3}\\
\quad\quad\quad\quad\quad\quad\quad +...3,\ \&c.
\end{array}
$$

i. e. $1+\frac{1}{2}+\frac{1}{3}$, divided by $1-\frac{1}{2}$, produces the quotient $1+\frac{2}{3}+\frac{3}{6}+\frac{3}{10}+\frac{3}{15}$, &c. For the terms of the dividend are $1+\frac{1}{2}+\frac{1}{3}$, and not $1+\frac{1}{3}+\frac{1}{6}$, because the number of them is finite; but in the quotient, the number of terms being infinite, the value of the terms consists in their position with reference to unity. This being premised, the sum of the terms of the dividend is $\frac{11}{6}$, and this, divided by $\frac{1}{2}$, gives $\frac{22}{6}$. Again, the sum of the terms $1+\frac{2}{3}+\frac{3}{6}$ of the quotient is $\frac{13}{6}$; and this added to the remainder $\frac{3}{4}$, divided by $\frac{1}{2}$, which is the divisor, *i. e.* added to $\frac{6}{4}$, is $\frac{13}{6}+\frac{6}{4}=\frac{88}{24}=\frac{22}{6}$.

$$
\begin{array}{l}
1-.1)\ 1+.1+..1+...1\ (1+.2+..3+...4+....4, \&c. \\
\quad \underline{1-.1} \\
\qquad +.2+..1 \\
\qquad \underline{+.2-..2} \\
\qquad\qquad +..3+...1 \\
\qquad\qquad \underline{+..3-...3} \\
\qquad\qquad\qquad +...4, \&c.
\end{array}
$$

i. e. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$, divided by $1-\frac{1}{2}$, produces $1+\frac{2}{3}+\frac{3}{6}+\frac{4}{10}+\frac{4}{15}$, &c. For $\frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{1-\frac{1}{2}}=\frac{100}{24}=4\frac{1}{6}$. And $1+\frac{2}{3}+\frac{3}{6}+\frac{13}{6}$; but this added to the remainder $\frac{4}{4}$, divided by $\frac{1}{2}$, *i. e.* added to 2, will be $\frac{13}{6}+\frac{2}{1}=\frac{25}{6}=4\frac{1}{6}$. Thus, too, $\frac{1+.1+..1+...1+....1}{1-.1}=\frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}}{1-\frac{1}{2}}=\frac{548}{120}$. And the quotient $1+.2+..3+...4+....5+.....5$, &c. $=1+\frac{2}{3}+\frac{3}{6}+\frac{4}{10}+\frac{5}{15}+\frac{5}{21}$, &c. $=\frac{274}{60}=\frac{548}{120}$. And this will be the case universally, when the dividend is any other number of the terms of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c. *ad infinitum.*

Again, $\frac{1+.\frac{1}{2}+..\frac{1}{3}+...\frac{1}{4}}{1-.1}=\frac{1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}}{1-\frac{1}{2}}$ is, when expanded, the series $1+.\frac{3}{2}+..\frac{11}{6}+...\frac{50}{24}+....\frac{50}{24}+.....\frac{50}{24}$, &c. $=1+\frac{3}{6}+\frac{11}{66}+\frac{50}{240}+\frac{50}{750}+\frac{50}{1050}$, &c. $=\frac{205}{72}$. And $\frac{1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}}{1-\frac{1}{2}}=\frac{410}{144}=\frac{205}{72}$. Thus, too, $\frac{1+.\frac{1}{2}+..\frac{1}{4}}{1-.1}=\frac{1+\frac{1}{4}+\frac{1}{12}}{1-\frac{1}{2}}$; and this expanded will be $1+.\frac{3}{2}+..\frac{7}{4}+...\frac{7}{4}+....\frac{7}{4}$, &c. $=$

$1+\frac{3}{6}+\frac{7}{24}+\frac{7}{40}+\frac{7}{60}$, &c. $=\frac{128}{48}=\frac{8}{3}$. And $\frac{1+\frac{1}{4}+\frac{1}{12}}{1-\frac{1}{2}}$ $=\frac{4}{3}$, divided by $\frac{1}{2}=\frac{8}{3}$.

In a similar manner it will be found, that $\frac{2-.1+..1-...1+....1}{1-.1}=2+.1+..2+...1$, &c. $=\frac{19}{6}$. Likewise that $\frac{2-..1}{1-.1}=2+.2+..1+...1$ $+....1$, &c. $=\frac{10}{3}$. And that $\frac{1-...\frac{1}{2}}{1-.1}=1+.1$ $+..1+...\frac{1}{2}+....\frac{1}{2}+.....\frac{1}{2}$, &c. $=\frac{7}{4}$. Also that $\frac{1-.\frac{1}{2}-..\frac{1}{4}}{1-.1}=1+.\frac{1}{2}+..\frac{1}{4}+...\frac{1}{4}+....\frac{1}{4}$, &c. $=\frac{4}{3}$. And that $\frac{1-.\frac{1}{3}-..\frac{1}{9}}{1-.1}=1+.\frac{2}{3}+..\frac{5}{9}$ $+...\frac{5}{9}+....\frac{5}{9}$, &c. $=\frac{43}{27}$. Again, $\frac{1-.\frac{1}{2}-..\frac{1}{3}}{1-.1}$ $=1+.\frac{1}{2}+..\frac{1}{6}+...\frac{1}{6}+....\frac{1}{6}$, &c. $=\frac{23}{18}$. And, $\frac{1-.\frac{1}{2}-..\frac{1}{3}-...\frac{1}{4}}{1-.1}=1+.\frac{1}{2}+..\frac{1}{6}-...\frac{1}{12}-....\frac{1}{12}$, &c. $=\frac{83}{72}$. Likewise, $\frac{1-.\frac{1}{2}-..\frac{1}{4}-...\frac{1}{8}}{1-.1}=1+.\frac{1}{2}$ $+..\frac{1}{4}+...\frac{1}{8}=+....\frac{1}{8}=\frac{61}{48}$. And, $\frac{1+.\frac{3}{2}+..\frac{7}{4}}{1-.1}$ $=1+.\frac{5}{2}+..\frac{17}{4}+...\frac{17}{4}$, &c. $=\frac{14}{3}$. Also, $\frac{1+.2+..3}{1-.1}=1+.3+..6+...6$, &c. $=6$. And, $\frac{1+.2+..3+...4}{1-.1}=1+.3+..6+...10+$ $....10$, &c. $=8$. Farther still, $\frac{1+.3+..6}{1-.1}=1+.4$

$+ \,.\,.\, 10 + \,.\,.\,.\, 10 + \,.\,.\,.\,.\, 10$, &c. $= 9$. And, $\frac{1 +.\, 3 +..\, 6 +...\, 10}{1 -.\, 1} = 1 +.\, 4 +..\, 10 +...\, 20 +$ $....\, 20$, &c. $= 14$.

9. In most cases it will be more convenient to substitute numbers for the points in this notation; viz. instead of $1 +.\, 1 +..\, 1 +...\, 1 +....\, 1 +.....\, 1 +......\, 1$, &c. to substitute $1 + {}^{1}1 + {}^{2}1 + {}^{3}1 + {}^{4}1 + {}^{5}1 + {}^{6}1$, &c. And so in other instances.

11. Since, by the seventh proposition of the tenth book of Euclid, incommensurable quantities have not that proportion to each other which number has to number; and since, in all division of commensurable quantities, the terms of the quotient, however numerous they may be, are, when added to each other, and multiplied by the divisor, equal to the dividend (an infinitesimal excepted, when the terms of the quotient form an infinite series): hence all such infinite series as are produced by the expansion of expressions, to which the aggregates of the terms severally added to each other are not equal, are incommensurable quantities. And that there are such infinite series is evident, in our method of notation, from the following instances:—

$$\begin{array}{l} 1-\frac{1}{3} \\ 1-..1 \end{array}\Big)\ 1\ \Big(\begin{array}{l} 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25},\ \&c. \\ 1+{}^{2}1+{}^{4}1+{}^{6}1+{}^{8}1,\ \&c. \end{array}$$

This series, as modern mathematicians have demonstrated, is one sixth part of the square of the circumference of a circle whose diameter is unity; and the aggregate of it is evidently greater than the value of the expression $\frac{1}{1-..1}$, or $\frac{1}{1-\frac{1}{3}}=\frac{3}{2}$. For, if this was the true value of its aggregate, the circumference itself would be only equal to 3. For, $6\times\frac{3}{2}=\frac{18}{2}=9,=$ the square of the circumference, and the square root of 9 is 3, the circumference itself. Hence the series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}$, &c. is incommensurable, and therefore six times this series, or the square of the circumference, is incommensurable; and, consequently, the circumference itself is incommensurable.

Thus, too, $\frac{1}{1-{}^{3}1}$ expanded, will be the series $1+{}^{3}1+{}^{6}1+{}^{9}1+{}^{12}1$, &c.; *i. e.* $1+\frac{1}{5}+\frac{1}{12}+\frac{1}{22}+\frac{1}{35}$, &c.; being the reciprocal of the pentagonal series $1+5+12+22+35$, &c.; and will be incommensurable. For the aggregate of the series will exceed $\frac{1}{1-{}^{3}1}$ or $\frac{1}{1-\frac{1}{4}}$; *i. e.* $\frac{4}{3}$. This likewise will be the case with all the reciprocals of the other polygonous series, *i. e.* with $\frac{1}{1-{}^{4}1}$, or $1+{}^{4}1+{}^{8}1$

$+ {}^{12}1 + {}^{16}1$, &c. $= 1 + \frac{1}{6} + \frac{1}{15} + \frac{1}{28} + \frac{1}{45}$, &c. being the reciprocal of the hexagonal series of whole numbers. And also with $\frac{1}{1 - {}^{5}1}$, or $1 + {}^{5}1 + {}^{10}1 + {}^{15}1 + {}^{20}1$, &c., which is the reciprocal of the heptagonal series $1 + 7 + 18 + 34 + 55$, &c. And so in all the rest of these series. Hence it follows, that the aggregates of all the polygonous reciprocal series, except the triangular, are incommensurable quantities.

This will likewise be the case with the series $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11}$, &c., $1 + \frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \frac{1}{13}$, &c., $1 + \frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \frac{1}{17} + \frac{1}{21}$, &c., and all other such like series, which are parts of the infinite series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$, &c. For they will be the incommensurable half, third, fourth, &c. parts of that series. This will be at once evident, by considering that the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10}$, &c. $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \frac{1}{15}$, &c., $\frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \frac{1}{16} + \frac{1}{20}$, &c., are the commensurable half, third, fourth, &c. parts of the said series; because the aggregate of any number of the terms of these series will be the $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. parts of the like number of terms of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$, &c., which will not be the case with the parts of the other series.

And from what we have shown, (in parag. 5), it is evident that the series $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}$, &c. $1+\frac{1}{4}+\frac{1}{7}+\frac{1}{10}$, &c. $1+\frac{1}{5}+\frac{1}{9}+\frac{1}{13}$, &c., differ from the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}$, &c. $\frac{1}{3}+\frac{1}{6}+\frac{1}{9}$, &c. $\frac{1}{4}+\frac{1}{8}+\frac{1}{12}$, &c. by fractional parts of unity. The expressions, however, from which both these series are expanded, are equal: and hence the former are incommensurable.

12. According to our notation, $.1$, $..1$, $...1$, &c., signify, in multiplication, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. in infinite, but not in finite series. Hence the true value of the expression $\frac{1}{1-.2+..1}$, when expanded, will be $1+\frac{2}{2}+\frac{3}{4}+\frac{4}{8}+\frac{5}{16}$, &c.; and not $1+\frac{2}{3}+\frac{3}{6}+\frac{4}{10}+\frac{5}{15}$, &c., because this expression is the product of $\frac{1}{1-.1}\times\frac{1}{1-.1}$, and this product being finite, is equivalent to $\frac{1}{1-1+\frac{1}{4}}$. And so in other instances.

13. The reciprocal of the hexagonal series is produced, as we have already observed, by the expansion of the expression $\frac{1}{1-{}'1}=\frac{5}{4}=1+\frac{1}{6}+\frac{1}{15}+\frac{1}{28}$, &c., and, consequently, the series $\frac{1}{3}+\frac{1}{10}+\frac{1}{21}+\frac{1}{36}$, &c. is equal to $\frac{3}{4}$: for the aggregate of both these series is equal to $2=1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}$, &c. But the series $1+\frac{1}{6}+\frac{1}{15}+\frac{1}{28}$,

&c., as we have observed (in parag. 11), is incommensurable; and therefore the half of it, or the series $\frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \frac{1}{56}$, &c., discovered by Lord Brouncker for the quadrature of the hyperbola, is also incommensurable.

14. (In parag. 11) we have shown, that in consequence of the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$, &c. being incommensurable, the circumference of the circle is also incommensurable with its diameter. But that it is so, is likewise evident from the infinite series discovered by Leibnitz, the aggregate of which is equal to the area of a circle whose diameter is 1. For this series is an expansion of the expression $\frac{1 +. 1}{1 + 1}$, *i. e.* is $1 -. 1^{\frac{1}{3}} +. 1^{\frac{1}{5}} -. 1^{\frac{1}{7}} +. 1^{\frac{1}{9}} -. 1^{\frac{1}{11}}$, &c. But the expression $\frac{1 +. 1}{1 + 1}$, or $\frac{1 + \frac{1}{2}}{1 + 1} = \frac{3}{4}$; and four times this will give the circumference, which will therefore only be $\frac{12}{4} = 3$; and, consequently, the aggregate of this series is greater than the expression, by the expansion of which it is produced, and is therefore incommensurable. At the same time, however, we may see that what modern mathematicians have asserted respecting the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$, &c., viz. that it is $\frac{1}{6}$ of the square of the circumference,

follows from the expressions by which that series, and the series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}$, &c., are produced according to our method of notation.

15. Since the expression $\frac{1}{1-.1}$, is most evidently equivalent when expanded to the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c., it follows that the expression $\frac{1}{1+.1}$ must, when expanded, be $1-\frac{1}{3}+\frac{1}{6}-\frac{1}{10}+\frac{1}{15}-\frac{1}{21}$, &c.; just as because $\frac{1}{2-1}$ is, when expanded, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, &c. it necessarily follows that $\frac{1}{2+1}$ is, when expanded, $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}-\frac{1}{64}$, &c. But, as $1+.1$ is $\frac{1}{1+\frac{1}{2}}=\frac{2}{3}$, it is evident that the series $1-\frac{1}{3}+\frac{1}{6}-\frac{1}{10}$, &c., is one of those series of which the value of the expression from which it is produced is less than the aggregate value of the terms severally taken, and, consequently, that it is conformably to what we have before observed, an incommensurable series. For the value of the first term of this series, minus the second, is at once equal to $\frac{2}{3}$.

16. From our method of notation, every term in an infinite series has a twofold value, when the points by which the terms are designated are in an arithmetical progression; and one of

these values arises from multiplication, but the other from position; the aggregate of each series being sometimes equal, and sometimes not. Thus, for instance, $\frac{1}{1-.1}$, is, when expanded, $1+.1+{}^{2}1+{}^{3}1+{}^{4}1$, &c. Here, as the divisor is equivalent to $1-\frac{1}{2}$, the value of the quotient, as produced by the division of 1 by $1-\frac{1}{2}$, will be $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, &c. For, in this case, the third term, or ${}^{2}1$, will be $\frac{1}{4}$, because produced by the multiplication of $.1$, or $\frac{1}{2}$, into itself; since $.1 \times .1$ will then be equal to $..1$ or $\frac{1}{4}$. Thus, too, the fourth term, or ${}^{3}1$, is equivalent to $\frac{1}{8}$. For the second term ${}^{2}1$, multiplied by $.1$, produces ${}^{3}1$, *i. e.* $\frac{1}{8}$. And so of the rest. From position, however, the second term $.1$ is $\frac{1}{3}$, the third term ${}^{2}1$ is $\frac{1}{6}$, the fourth term ${}^{3}1$ is $\frac{1}{10}$, the fifth term ${}^{4}1$ is $\frac{1}{15}$, and so of the rest. And the aggregate of $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, &c., is equal to the aggregate of $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c. But if $\frac{1}{1-{}^{2}1}=\frac{3}{2}$ is expanded, the series will be $1+{}^{2}1+{}^{4}1+{}^{6}1+{}^{8}1$, &c. And the value of the terms arising from multiplication will be $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}$, &c.; but from position $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}$, &c., which may be demonstrated in the same manner as

in the preceding instance. Though both the series, however, are expansions of the expression $\frac{1}{1-{}^{2}1}$, yet the former is a commensurable, but the latter an incommensurable series. And the like will take place in all the other reciprocals of polygonous series.

17. That when $.1, ..1, ...1$, &c. *i. e.* $^{1}1$, $^{2}1$, $^{3}1$, &c. are the numerators of fractions, in our method of notation, the place of unity in the quotient is not in the first point, but immediately before it, may be thus demonstrated. Let the expression be $\frac{.1}{1-.1}=\frac{\frac{1}{2}}{1-\frac{1}{2}}$, this, when expanded, will be $.1+{}^{2}1+{}^{3}1+{}^{4}1$, &c. But instead of $.1$ let its equal $\frac{1}{2}$ be the numerator, and then the series will be $\frac{1}{2}+.\frac{1}{2}+{}^{2}\frac{1}{2}+{}^{3}\frac{1}{2}+{}^{4}\frac{1}{2}$, &c. $=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}$, &c., which is the true value, and is equal to 1. Hence $.1+{}^{2}1+{}^{3}1+{}^{4}1$, &c. will be, $\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c., equal to the aggregate of the former series; *i. e.* equal to 1. The place of unity, therefore, is not in the first point of the series $.1+{}^{2}1+{}^{3}1+{}^{4}1$, &c., but immediately before it: for, if it were, this series would become $\frac{1}{2}+\frac{1}{5}+\frac{1}{9}+\frac{1}{14}$, &c., the value of which is evidently

greater than 1, and is therefore greater than its true value.

This notation is analogous to that of decimals. For thus in decimals, 5 . 4, or 5, 4, is equivalent to 5 and $\frac{4}{10}$ of 1, the place of unity not being in the point, but immediately preceding it.

18. In geometrical series, such as $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, &c., $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}$, &c., $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}$, &c., the method of proceeding, according to our notation, is as follows: —

$$
\begin{array}{l}
2-1\,)\ 1\qquad\Big(0+\overset{\frac{1}{2}}{1}+.\overset{\frac{1}{4}}{1}+{}^{3}\overset{\frac{1}{8}}{1}+{}^{7}\overset{\frac{1}{16}}{1}+{}^{15}\overset{\frac{1}{32}}{1},\ \&c.\\
\qquad\ \ \underline{1-1}\\
\qquad\quad +1\\
\qquad\quad \underline{+1-.1}\\
\qquad\qquad\quad +.1\\
\qquad\qquad\quad \underline{+.1-{}^{3}1}\\
\qquad\qquad\qquad\quad +{}^{3}1\\
\qquad\qquad\qquad\quad \underline{+{}^{3}1-{}^{7}1}\\
\qquad\qquad\qquad\qquad\quad +{}^{7}1,\ \&c.
\end{array}
$$

Here, in the first place, 1 divided by 2 is $\frac{1}{2}$, which in the quotient is 0+1; and the quotient $\frac{1}{2}$ multiplied by the divisor 2−1 will be 1−$\frac{1}{2}$, or 1−. 1. But this, with reference to the position of the terms of the quotient, will be 1−1. Then 1−1, subtracted from 1, will leave +1; and this divided by 2, will be $\frac{1}{2}$ or . 1; which placed in

the quotient, will become from position $\frac{1}{4}$, as is evident from inspection. Again, this .1 multiplied by the divisor 2—1, will produce 1—.1. But this subtracted from 1 will leave +.1, which, divided by 2, will give .1 or $\frac{1}{4}$. And this, from position in the quotient, will become $\frac{1}{8}$. And so of the rest of the terms.

This method of proceeding may be illustrated by the mode of division in decimal arithmetic, to which it is analogous. Thus, for instance, in producing the decimal of $\frac{1}{9}$, the process is as follows:—

```
9) 1000000, &c. (,11111, &c.
   .9
   ---
   .10
    .9
   ----
    .10
     .9
    ----
    .10, &c.
```

Here the first term of the quotient is ,1 or 01, which from position with reference to unity is $\frac{1}{10}$, and this multiplied by the divisor 9 becomes $\frac{9}{10}$; but the first subtracter 9, though it is in reality from position $\frac{9}{10}$, is subtracted from the dividend as 9: just as in the above instance the

-1 of the first subtracter is in reality, from position, $-\frac{1}{3}$, but is subtracted as minus unity. And it is obvious that the rest of the division in the former, will be analogous to that in the latter instance.

Thus, too, in the triple geometrical series $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}$, &c., the process will be as follows: —

$$
\begin{array}{l}
3-1)\ 1\ (0+.\overset{\frac{1}{3}}{1}+{}^{5}\overset{\frac{1}{9}}{1}+{}^{17}\overset{\frac{1}{27}}{1}+{}^{53}\overset{\frac{1}{81}}{1},\ \&c. \\
\quad\ \ \underline{1-.1} \\
\qquad\ +.1 \\
\qquad\ \underline{+.1-{}^{5}1} \\
\qquad\qquad\ +{}^{5}1 \\
\qquad\qquad\ \underline{+{}^{5}1-{}^{17}1} \\
\qquad\qquad\qquad\ +{}^{17}1,\ \&c.
\end{array}
$$

In this instance, in the first place 1 divided by 3 is $\frac{1}{3}$, which in the quotient is $0+.1$; and this quotient multiplied by the divisor $3-1$, will be $1-\frac{1}{3}$ or $1-..1$. But this, with reference to the position of the terms of the quotient, will be $1-.1$. Then $1-.1$ subtracted from 1 will leave $+.1$, and this divided by 3 will be $\frac{1}{9}$ or ${}^{5}1$, which placed in the quotient, will become from position $\frac{1}{9}$. Again, this ${}^{5}1$ multiplied by the divisor $3-1$, will produce $.1-{}^{5}1$. But this, by subtraction, will leave $+{}^{5}1$, which divided

by 3 will give ${}^{17}1$ or $\frac{1}{18}$. And this, from position in the quotient, will be $\frac{1}{27}$. And so of the other terms.

In a similar manner, also, the quadruple series $\frac{1}{4-1}$, when expanded, will be as below:—

$$
4-1)\begin{array}{l}
1 \\
\underline{1-{}^{3}1} \\
\quad +{}^{3}1 \\
\quad \underline{+{}^{3}1-{}^{11}1} \\
\qquad\quad +{}^{11}1 \\
\qquad\quad \underline{+{}^{11}1-{}^{47}1} \\
\qquad\qquad\quad +{}^{47}1 \\
\qquad\qquad\quad \underline{+{}^{47}1-{}^{191}1} \\
\qquad\qquad\qquad\quad +{}^{191}1,\ \&c.
\end{array}
\left(0+\overset{\frac{1}{4}}{{}^{3}1}+\overset{\frac{1}{16}}{{}^{11}1}+\overset{\frac{1}{64}}{{}^{47}1}+\overset{\frac{1}{256}}{{}^{191}1},\ \&c.\right.
$$

And so in all other fractional geometrical series.

The following also are examples of this method of notation in geometrical series:—

$$
2-\tfrac{1}{2})\begin{array}{l}
1 \\
\underline{1-{}^{3}1} \\
\quad +{}^{3}1 \\
\quad \underline{+{}^{3}1-{}^{11}1} \\
\qquad\quad +{}^{11}1,\ \&c.
\end{array}
\left(0+\overset{\frac{1}{2}}{1}+\overset{\frac{1}{8}}{{}^{5}1}+\overset{\frac{1}{32}}{{}^{23}1},\ \&c.\right.
$$

$$
3-\tfrac{1}{3})\begin{array}{l}
1 \\
\underline{1-{}^{7}1} \\
\quad +{}^{7}1 \\
\quad \underline{+{}^{7}1-{}^{71}1} \\
\qquad\quad +{}^{71}1 \\
\qquad\quad \underline{+{}^{71}1-{}^{647}1} \\
\qquad\qquad\quad +{}^{647}1,\ \&c.
\end{array}
\left(0+.\ \overset{\frac{1}{3}}{1}+\overset{\frac{1}{24}}{{}^{23}1}+\overset{\frac{1}{216}}{{}^{215}1}+\overset{\frac{1}{1944}}{{}^{1943}1},\ \&c.\right.
$$

$$
\begin{array}{l}
4-1)\ \tfrac{2}{3}\ \ (0+{}^{4}1+{}^{17}1+{}^{71}1+{}^{287}1, \&c. \\
\quad \underline{\tfrac{2}{3}-{}^{5}1} \\
\qquad +{}^{5}1 \\
\qquad \underline{+{}^{5}1-{}^{17}1} \\
\qquad\qquad +{}^{17}1 \\
\qquad\qquad \underline{+{}^{17}1-{}^{71}1} \\
\qquad\qquad\qquad +{}^{71}1 \\
\qquad\qquad\qquad \underline{+{}^{71}1-{}^{287}1} \\
\qquad\qquad\qquad\qquad +{}^{287}1, \&c.
\end{array}
$$

19. The series $0+\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}$, &c., if each of the denominators is multiplied by $\frac{1}{2}$, will become $0+\frac{1}{4}+\frac{1}{8}+\frac{1}{12}+\frac{1}{16}+\frac{1}{20}$, &c.; and this by position, according to our method, will be $0+{}^{4}1+{}^{8}1+{}^{12}1+{}^{16}1+{}^{20}1$, &c., when the position of the terms is considered with reference to the denominators, and not to the value of the several fractions. Hence it will be, by position, equal to $\frac{1}{6}+\frac{1}{15}+\frac{1}{28}+\frac{1}{45}+\frac{1}{66}$, &c.; which series, as we have shown in (parag. 11), is equal to $\frac{1}{4}$ incommensurably. And as the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}$, &c. is the half of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c., so $\frac{1}{4}+\frac{1}{8}+\frac{1}{12}+\frac{1}{16}+\frac{1}{20}$, &c. is the $\frac{1}{4}$ of the same series. Thus, too, one third of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c., will be $\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}$, &c., and this multiplied by $\frac{1}{2}$ will be $\frac{1}{6}+\frac{1}{12}+\frac{1}{18}+\frac{1}{24}+\frac{1}{30}$, &c., and this, by position, will be $0+{}^{6}1+{}^{12}1+{}^{18}1+{}^{24}1+{}^{30}1$, &c. $=\frac{1}{8}+\frac{1}{21}+\frac{1}{40}+\frac{1}{65}+\frac{1}{96}$, &c. $=\frac{1}{6}$ incommensurably. For

$\frac{1}{1-{}^{6}1} = 1 + {}^{6}1 + {}^{12}1 + {}^{18}1 + {}^{24}1$, &c. $= \frac{1}{1-\frac{1}{7}} = \frac{7}{6} = 1\frac{1}{6}$. And therefore $\frac{1}{8} + \frac{1}{21} + \frac{1}{40}$, &c. $= \frac{1}{6}$. In like manner one third of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$, &c., viz. $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \frac{1}{15}$, &c., will become by position, if 0 is put in the place of unity, $0 + {}^{3}1 + {}^{6}1 + {}^{9}1 + {}^{12}1 + {}^{15}1$, &c. $= \frac{1}{5} + \frac{1}{12} + \frac{1}{22} + \frac{1}{35} + \frac{1}{51}$, &c., equal to the series which is the reciprocal of the pentagonal series of whole numbers $5 + 12 + 22 + 35 + 51$, &c., and this reciprocal series is equal to $\frac{1}{4}$ incommensurably. For

$$\begin{array}{llll}
1-{}^{3}1\,) & 1 & (& 1+{}^{3}1+{}^{6}1+{}^{9}1, \text{ \&c.} = 1+\frac{1}{5}+\frac{1}{12}+\frac{1}{22}, \text{ \&c.} \\
 & & & = \frac{3}{4})\ 1\ (= \frac{4}{3} = 1\frac{1}{3}. \\
 & 1-{}^{3}1 & & \\
\hline
 & +{}^{3}1 & & \\
 & +{}^{3}1-{}^{6}1 & & \\
\hline
 & \quad +{}^{6}1 & & \\
 & \quad +{}^{6}1-{}^{9}1 & & \\
\hline
 & \qquad +{}^{9}1, \text{ \&c.} & &
\end{array}$$

Again, one fourth of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$, &c. will be $\frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \frac{1}{16} + \frac{1}{20} + \frac{1}{24}$, &c., and this, by position as above, will be equal to $\frac{1}{6} + \frac{1}{15} + \frac{1}{28} + \frac{1}{45} + \frac{1}{66}$, &c. $= \frac{1}{4}$ incommensurably, and will be the reciprocal of the hexagonal series of whole numbers $6 + 15 + 28 + 45$, &c.

20. The series $1 + {}^{2}1 + {}^{5}1 + {}^{9}1 + {}^{14}1$, &c., by our method of notation, when the position of the terms with reference to unity is not considered,

will be $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c.; but if the terms are considered with reference to unity, the series will become $1+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}$, &c. But the aggregate of $\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c. is 1, and 1 by position, with reference to a unity immediately preceding it, will be $\frac{1}{2}$; so that the series $1+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}$, &c. will be equal to $1\frac{1}{2}$. Thus also the series $1+\frac{1}{5}+\frac{1}{15}+\frac{1}{35}+\frac{1}{70}$, &c. will be equal to $1\frac{1}{3}$. For $1+{}^{3}1+{}^{9}1+{}^{19}1$, &c., when the position of the terms with reference to unity is not considered, will be $1+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}$, &c.; but if the terms are considered with reference to unity, the series will be equal to $1+\frac{1}{5}+\frac{1}{15}+\frac{1}{35}+\frac{1}{70}$, &c. But the aggregate of $\frac{1}{4}+\frac{1}{10}+\frac{1}{20}$, &c. is $\frac{1}{2}$, and $\frac{1}{2}$ by position with reference to unity, will be equal to $\frac{1}{3}$, so that the series $1+\frac{1}{5}+\frac{1}{15}+\frac{1}{35}$, &c. will be equal to $1\frac{1}{3}$. And Dobson, in his Mathematical Repository, has demonstrated that the former of these series is equal to $\frac{3}{2}$, and the latter to $\frac{4}{3}$. By the above method, however, the aggregates of the reciprocals of all the other infinite series of figurate numbers may be easily obtained. For thus it will be found that the sum of the series $1+\frac{1}{6}+\frac{1}{21}+\frac{1}{56}+\frac{1}{126}$, &c. is equal to $\frac{5}{4}$. For the aggregate of $\frac{1}{5}+\frac{1}{15}+\frac{1}{35}+\frac{1}{70}$, &c. is $\frac{1}{3}$, and $\frac{1}{3}$ by

position, with reference to unity, is $\frac{1}{4}$. And thus it will be found that the fractional aggregates of the reciprocals of all the figurate numbers, will be 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, &c. *ad infinitum;* which may also be demonstrated by Dobson's method, though not with the same facility as by ours.

21. Hence, from what has been demonstrated in parag. 19, it follows that the series $\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\frac{1}{125}$, &c. is incommensurably equal to $\frac{1}{6}$. For ${}^{6}1+6\frac{1}{3}+6\frac{1}{6}+6\frac{1}{10}$, &c. is equal to the $\frac{1}{6}$ of $2=\frac{1}{3}$; and the half of this series, viz. . $+\frac{1}{7}+\frac{1}{19}+\frac{1}{37}+\frac{1}{61}$, &c., will be from position equal to $\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\frac{1}{125}$, &c. $=\frac{1}{6}$ incommensurably.

22. In the following instances, viz. $\frac{0+{}^{\cdot}1}{1-{}^{2}1}$, $\frac{0+{}^{2}1}{1-{}^{3}1}$, $\frac{0+{}^{3}1}{1-{}^{4}1}$, &c., the numerators will be equivalent to $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c.

$$
\begin{array}{l}
1-{}^{2}1)\ 0+{}^{\cdot}1\ (0+{}^{\cdot}1+{}^{3}1+{}^{5}1+{}^{7}1,\ \&c.=\frac{1}{3}+\frac{1}{7}+\frac{1}{13}+\frac{1}{21},\ \&c.\\
\qquad\ \ \underline{0+{}^{\cdot}1-{}^{3}1}\\
\qquad\qquad\quad +{}^{3}1\\
\qquad\qquad\quad \underline{+{}^{3}1-{}^{5}1}\\
\qquad\qquad\qquad\ \ +{}^{5}1\\
\qquad\qquad\qquad\ \ \underline{+{}^{5}1-{}^{7}1}\\
\qquad\qquad\qquad\qquad +{}^{7}1,\ \&c.
\end{array}
$$

$1-{}^{3}1)\ 0+{}^{2}1\ (0+{}^{2}1+{}^{5}1+{}^{8}1+{}^{11}1,\ \&c.=\frac{1}{4}+\frac{1}{10}+\frac{1}{19}+\frac{1}{31},\ \&c.$

And if the terms of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c, are severally multiplied by $\frac{1}{2}$, the series $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}$, &c.$=1$ will be produced, *i. e.* in our

notation, $.1 + {}^{3}1 + {}^{5}1 + {}^{7}1$, &c.; the half of which will be (from parag. 20) $.+.1 + {}^{3}1 + {}^{5}1 + {}^{7}1$, &c. $= \frac{1}{3} + \frac{1}{7} + \frac{1}{13} + \frac{1}{21}$, &c. $= \frac{1}{2}$, incommensurably. Thus, too, if the terms of the series $1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15}$, &c. are severally multiplied by $\frac{1}{3}$, the series $\frac{1}{3} + \frac{1}{9} + \frac{1}{18} + \frac{1}{30}$, &c. $= \frac{2}{3}$ will be produced, *i. e.* ${}^{2}1 + {}^{5}1 + {}^{8}1 + {}^{11}1$, &c.; the half of which will be $.+{}^{2}1 + {}^{5}1 + {}^{8}1 + {}^{11}1$, &c. $= \frac{1}{4} + \frac{1}{10} + \frac{1}{19} + \frac{1}{31}$, &c. $= \frac{2}{6} = \frac{1}{3}$. And $\frac{0+.1}{1-{}^{2}1} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{2}{3}) \frac{1}{3} (= \frac{3}{6} = \frac{1}{2}$. Thus, too, $\frac{0+{}^{3}1}{1-{}^{3}1} = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{3}{4}) \frac{1}{4} (= \frac{4}{12} = \frac{1}{3}$. And so in all other instances.

23. As the series $1 + .1 + {}^{2}1 + {}^{3}1 + {}^{4}1$, &c. *i. e.* $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$, &c. produces, from position, the series $1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21}$, &c.; so the half of this series, viz. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9}$, &c., produces the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$, &c.; the one third of it, viz. $1 + \frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \frac{1}{13}$, &c. produces the series $1 + \frac{1}{5} + \frac{1}{12} + \frac{1}{22} + \frac{1}{35}$, &c.; and in a similar manner the remaining parts of it produce the remaining series, the reciprocals of polygonous series. And as the series $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9}$, &c. is the half of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$, &c., so the series $\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$, &c. is the half of the series $\frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21}$, &c. And, uni-

versally, the $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c. parts of the former will produce the $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, &c. parts of the latter series.

24. Again, the other half of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, &c., viz. the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}$, &c. will produce, by position, the series $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}$, &c., which is the half of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c.; and the other $\frac{1}{3}$ of that series, viz. $\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}$, &c. will produce, by position, the series $\frac{1}{3}+\frac{1}{9}+\frac{1}{18}+\frac{1}{30}+\frac{1}{45}$, &c., the $\frac{1}{3}$ of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c. And so on.

25. As the parts of the infinite series likewise $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$, &c. in parag. 23, are incommensurable, so likewise are the parts of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c. in the same paragraph. And as the parts of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$, &c. in parag. 24, are commensurable, so also are the parts in the same paragraph of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c.

26. The terms of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c., when they are considered as to their position with reference to unity, form the series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}$, &c. For $\frac{1}{2}$ is, by position, . $\frac{1}{2}$, *i. e.* $\frac{1}{4}$; $\frac{1}{3}$ is, by position . . $\frac{1}{3}$, *i. e.* $\frac{1}{9}$, and so of the rest of the terms. In like manner the series $1+.\frac{1}{2}+.\frac{1}{3}$

$+ \cdot \frac{1}{4} + \cdot \frac{1}{5} + \cdot \frac{1}{6}$, &c. will form the series $1 + \frac{1}{6} + \frac{1}{15} + \frac{1}{28} + \frac{1}{45} + \frac{1}{66}$, &c. For $1 + \cdot \frac{1}{2}$ will be, from position, $1 + \frac{1}{3} \times \frac{1}{2} = 1 + \frac{1}{6}$. And $\frac{1}{3}$, from its position, will be equivalent to $\frac{1}{5} \times \frac{1}{3} = \frac{1}{15}$; for it is in the fifth place from unity. And so of the rest of the terms. And as the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$, &c., is incommensurable, and is produced by the expansion of an expression equal to $\frac{4}{3}$, so likewise the series $1 + \frac{1}{6} + \frac{1}{15} + \frac{1}{28}$, &c. is incommensurable, and is produced by the expansion of an expression equal to $\frac{5}{4}$. For it is produced as below: —

$$
\begin{array}{l}
1 - {}^{4}1 \,)\; 1 \;\left(1 + \overset{\frac{1}{6}}{{}^{4}1} + \overset{\frac{1}{15}}{{}^{8}1} + \overset{\frac{1}{28}}{{}^{12}1}, \text{ \&c.}\right. \\
\qquad\quad \underline{1 - {}^{4}1} \\
\qquad\qquad + {}^{4}1 \\
\qquad\qquad \underline{+ {}^{4}1 - {}^{8}1} \\
\qquad\qquad\qquad + {}^{8}1 \\
\qquad\qquad\qquad \underline{+ {}^{8}1 - {}^{12}1} \\
\qquad\qquad\qquad\qquad + {}^{12}1, \text{ \&c.}
\end{array}
$$

Hence, as $0 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$, &c. produces, by position, the series $\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$, &c., so the half of that series, viz. $0 + \cdot \frac{1}{2} + \cdot \frac{1}{3} + \cdot \frac{1}{4} + \cdot \frac{1}{5}$, &c. will produce, by position, a series the half of $\frac{1}{4} + \frac{1}{9} + \frac{1}{16}$, &c., viz. it will produce the series $\frac{1}{6} + \frac{1}{15} + \frac{1}{28}$, &c.

27. The series $\frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \frac{1}{6 \cdot 8} + \frac{1}{8 \cdot 10}$, &c. $= \frac{1}{8} + \frac{1}{24} + \frac{1}{48} + \frac{1}{80}$, &c., will be, according to our

notation, $0+{}^{6}1+{}^{15}1+{}^{23}1+{}^{31}1$, &c., and the first term added to ${}^{15}1$, the second term in this notation, viz. $\frac{1}{8}+\frac{1}{16}$, will be equal to $\frac{3}{16}$, which is less than $\frac{1}{4}$ by $\frac{1}{16}$. In like manner the sum of the first and second terms, added to the third term in this notation, viz. $\frac{1}{8}+\frac{1}{24}+{}^{23}1=\frac{1}{12}+\frac{1}{8}=\frac{5}{24}$, which is less than $\frac{1}{4}$ by $\frac{1}{24}$. Thus, too, $\frac{1}{8}+\frac{1}{24}+\frac{1}{48}$, added to the fourth term in this notation, viz. ${}^{31}1$ or $\frac{1}{32}=\frac{7}{32}$, which is less than $\frac{1}{4}$ by $\frac{1}{32}$. And thus it will be found that the sum of the terms continually approximates to $\frac{1}{4}$, and will at last only differ from it by an infinitesimal. It is also remarkable, that the sum of any finite number of the terms always differs from $\frac{1}{4}$ by that term in our notation, which is added to the terms. Thus $\frac{1}{8}+\frac{1}{16}$ is less than $\frac{1}{4}$ by $\frac{1}{16}$. Thus, too, $\frac{1}{8}+\frac{1}{24}+\frac{1}{24}$ are less than $\frac{1}{4}$ by $\frac{1}{24}$. And $\frac{1}{8}+\frac{1}{24}+\frac{1}{48}+\frac{1}{32}$ are less than $\frac{1}{4}$ by $\frac{1}{32}$. And this will be found to be universally the case.

Again, the series $\frac{1}{2\,.\,5}+\frac{1}{5\,.\,8}+\frac{1}{8\,.\,11}+\frac{1}{11\,.\,14}$, &c. $=\frac{1}{10}+\frac{1}{40}+\frac{1}{88}+\frac{1}{154}$, &c., will be, in our notation, $0+{}^{8}1+{}^{29}1+{}^{47}1+{}^{65}1$, &c. And $\frac{1}{10}+{}^{29}1$, *i. e.* $\frac{1}{10}+\frac{1}{30}=\frac{4}{30}$, which is less than $\frac{1}{6}$ by $\frac{1}{30}$. Thus, too, $\frac{1}{10}+\frac{1}{40}+{}^{47}1$, or $\frac{1}{48}=\frac{7}{48}$, which is less than

$\frac{1}{6}$ by $\frac{1}{48}$. Likewise, $\frac{1}{10}+\frac{1}{40}+\frac{1}{88}+{}^{65}1$, or $\frac{1}{66}=\frac{10}{66}$, which is less than $\frac{1}{6}$ by $\frac{1}{66}$. And so in all other additions of the terms; by which it appears that the aggregate of this series will differ only from $\frac{1}{6}$ by an infinitesimal. In the same manner also, as in the preceding instance, the sum of any finite number of the terms always differs from $\frac{1}{6}$, by that term in our notation which is added to the terms.

Thus, too, the series $\frac{1}{3\,.\,6}+\frac{1}{6\,.\,9}+\frac{1}{9\,.\,12}+\frac{1}{12\,.\,15}$, &c. $=\frac{1}{18}+\frac{1}{54}+\frac{1}{108}+\frac{1}{180}$, &c., will be, in our notation, $0+{}^{16}1+{}^{35}1+{}^{53}1+{}^{71}1$, &c. And $\frac{1}{18}+\frac{1}{36}=\frac{3}{36}$, which is less than $\frac{1}{9}$ by $\frac{1}{36}$. Again, $\frac{1}{18}+\frac{1}{54}+\frac{1}{54}=\frac{5}{54}$, which is less than $\frac{1}{9}$ by $\frac{1}{54}$. And $\frac{1}{18}+\frac{1}{54}+\frac{1}{108}+\frac{1}{72}=\frac{7}{72}$, which is less than $\frac{1}{9}$ by $\frac{1}{72}$. And the like will take place in the summation of all the other terms; and also in other series of a similar formation.

Again, the series $\frac{1}{3\,.\,8}+\frac{1}{6\,.\,12}+\frac{1}{9\,.\,16}+\frac{1}{12\,.\,20}$ &c. $=\frac{1}{24}+\frac{1}{72}+\frac{1}{144}+\frac{1}{240}$, &c. $=\frac{1}{12}$, (see Bonnycastle's Algebra, p. 176,) will be, in our notation, $0+{}^{22}1+{}^{47}1+{}^{71}1+{}^{95}1$, &c. And $\frac{1}{24}+\frac{1}{48}=\frac{3}{48}$, which is less than $\frac{1}{12}$ by $\frac{1}{48}$. Thus, too, $\frac{1}{24}+\frac{1}{72}+\frac{1}{72}=\frac{5}{72}$, which is less than $\frac{1}{12}$ by $\frac{1}{72}$. And

this will take place in the summation of all the other terms; so that the terms by addition continually approximate to $\frac{1}{12}$, and will at length differ from it only by an infinitesimal.

28. In the geometrical series $\frac{1}{1\,.\,2}+\frac{1}{2\,.\,4}+\frac{1}{4\,.\,8}+\frac{1}{8\,.\,16}$, &c. $=\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}$, &c. $=\frac{2}{3}$, and in our notation $0+1+{}^{5}1+{}^{23}1+{}^{95}1$, &c.; if $\frac{1}{2}$ is added to $\frac{1}{6}$, the sum will be $\frac{8}{12}$, the aggregate of the series; for $\frac{8}{12}=\frac{2}{3}$. Thus, too, $\frac{1}{2}+\frac{1}{8}+\frac{1}{24}=\frac{16}{24}=\frac{2}{3}$. And $\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{96}=\frac{64}{96}=\frac{2}{3}$. And the like will take place in the summation of all the other terms.

Again, in the geometrical series $\frac{1}{1\,.\,3}+\frac{1}{3\,.\,9}+\frac{1}{9\,.\,27}+\frac{1}{27\,.\,81}+\frac{1}{81\,.\,243}$, &c. $=\frac{1}{3}+\frac{1}{27}+\frac{1}{243}+\frac{1}{2187}$, &c. $=\frac{3}{8}$, which, in our notation, will be $0+\ .\ 1+{}^{23}1+{}^{215}1+{}^{1943}1$, &c., if $\frac{1}{3}$ is added to $\frac{1}{24}$, the sum will be $\frac{9}{24}=\frac{3}{8}$. In like manner $\frac{1}{3}+\frac{1}{27}+\frac{1}{216}=\frac{81}{216}=\frac{3}{8}$. And thus it will be found that the aggregate of all the terms is $\frac{3}{8}$, an infinitesimal excepted. But that the series $\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}$, &c. is aggregately equal to $\frac{2}{3}$ is evident from this, that it is produced by the expansion of $\frac{1}{2-1\frac{1}{2}}$. And the series $\frac{1}{3}+\frac{1}{27}+\frac{1}{243}+\frac{1}{2187}$, &c.

is equal to $\frac{3}{8}$; because it is produced by the expansion of $\frac{1}{3-\frac{1}{3}}$. Thus, too, in the geometrical series $\frac{1}{1\,.\,4}+\frac{1}{4\,.\,16}+\frac{1}{16\,.\,64}+\frac{1}{64\,.\,256}$, &c. $=\frac{1}{4}+\frac{1}{64}+\frac{1}{1024}+\frac{1}{16384}$, &c. $=\frac{4}{15}$, and which, in our notation, will be $0+{}^{2}1+{}^{59}1+{}^{959}1+{}^{15359}1$, &c.; if $\frac{1}{4}$ is added to ${}^{59}1$, *i. e.* to $\frac{1}{60}$, the sum will be $\frac{16}{60}=\frac{8}{30}=\frac{4}{15}$. In like manner $\frac{1}{4}+\frac{1}{64}+{}^{959}1$, *i. e.*, $\frac{1}{960}=\frac{256}{960}=\frac{4}{15}$. And thus it will be found that the aggregate of all the terms is $\frac{4}{15}$. But that this is the sum of the series is evident: for the series is produced by the expansion of $\frac{1}{4-\frac{1}{4}}$. And it may easily be shown that the like property will take place in other similarly formed geometrical series, produced by the expansion of $\frac{1}{5-\frac{1}{5}}$, $\frac{1}{6-\frac{1}{6}}$, $\frac{1}{7-\frac{1}{7}}$, &c.

In the series $\frac{1}{3}+\frac{4}{9}+\frac{9}{27}+\frac{16}{81}+\frac{25}{243}$, &c. $=\frac{3}{2}$; (see Bonnycastle's Algebra, p. 174,) and which, in our notation, will be $0+.\,1+{}^{5}4+{}^{179}9+{}^{53}16$, &c.; if $\frac{1}{3}$ is added to ${}^{5}4$, *i. e.* to $\frac{4}{6}$, the sum will be 1, which is less than $\frac{3}{2}$ by $\frac{1}{2}=\frac{27}{54}$. Thus, too, $\frac{1}{3}+\frac{4}{9}+{}^{179}9$, *i. e.* to $\frac{9}{18}$, will be equal to $\frac{69}{54}$, which is less than $\frac{3}{2}$ by $\frac{12}{54}$. Again, $\frac{1}{3}+\frac{4}{9}+\frac{9}{27}+{}^{53}16$, or $\frac{16}{54}$, will be equal to $1\frac{22}{54}$, and $\frac{22}{54}$ is less than $\frac{1}{2}$ by $\frac{5}{54}$. And

so in other instances; the difference between the aggregate of the terms of the series and $\frac{3}{2}$ becoming at length infinitely small.

Again, the series $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\frac{6}{64}$, &c. $=2$, (see Bonnycastle's Algebra, p. 173,) will be, in our notation, $0+1+.\,2+{}^{3}3+{}^{7}4+{}^{15}5+{}^{31}6$, &c. Then $\frac{1}{2}+.\,2$, *i. e.* $\frac{1}{2}+1$, will be equal to $\frac{3}{2}$, which is less than 2 by $\frac{1}{2}$. In like manner $\frac{1}{2}+\frac{2}{4}+{}^{3}3$, *i. e.* to $\frac{3}{4}$, will be equal to $\frac{7}{4}$, which is less than 2 by $\frac{1}{4}$. Thus, too, $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+{}^{7}4$, or $\frac{4}{8}=\frac{15}{8}$, which is less than 2 by $\frac{1}{8}$. And thus it will be found, that there will be a continual approximation to 2 from the addition of the terms of the series, by the terms $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, &c., and, consequently, the difference between the aggregate of the terms and 2 will at length become infinitely small.

After the same manner it will be found that the series $\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\frac{4}{81}+\frac{5}{243}+\frac{6}{729}$, &c. will be aggregately equal to 1. For this series, in our notation, will be $0+.\,1+{}^{5}2+{}^{17}3+{}^{53}4+{}^{161}5+{}^{485}6$, &c. And $\frac{1}{3}+{}^{5}2$, or $\frac{4}{6}=\frac{2}{3}$, which is less than 1 by $\frac{1}{3}$, or $\frac{6}{18}$. Again, $\frac{1}{3}+\frac{2}{9}+{}^{17}3$, or $\frac{3}{18}=\frac{13}{18}$, which is less than 1 by $\frac{5}{18}$, or $\frac{15}{54}$. And $\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+{}^{53}4$,

or $\frac{4}{54}=\frac{40}{54}$, which is less than 1, by $\frac{14}{54}$. And so on; by which it appears that the difference between the aggregate of the terms of this series and 1, will at length be infinitely small.

Again, $\frac{1}{4}+\frac{2}{16}+\frac{3}{64}+\frac{4}{256}+\frac{5}{1024}+\frac{6}{4096}$, &c., will be, in our notation, $0+{}^{2}1+{}^{11}2+{}^{47}3+{}^{191}4+{}^{767}5+{}^{3071}6$, &c. And $\frac{1}{4}$ added to ${}^{11}2$, or $\frac{2}{12}=\frac{10}{24}=\frac{5}{12}$, which is less than $\frac{1}{2}$ by $\frac{1}{12}$. Again, $\frac{1}{4}+\frac{2}{16}+{}^{47}3$, or $\frac{3}{48}=\frac{7}{16}$, which is less than $\frac{1}{2}$ by $\frac{1}{16}$, or $\frac{12}{192}$. And $\frac{1}{4}+\frac{2}{16}+\frac{3}{64}+{}^{191}4$, or $\frac{4}{192}=\frac{85}{192}$, which is less than $\frac{1}{2}$ by $\frac{11}{192}$. And thus the difference between the aggregate of the terms of this series and $\frac{1}{2}$, will at length be an infinitesimal.

29. The series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$, &c. will be, in our notation, $0+1+.1+{}^{3}1+{}^{7}1+{}^{15}1$, &c. And if the third term in this notation, viz. $.1$ or $\frac{1}{2}$, is added to $\frac{1}{2}$, the first term; the sum 1 is equal to the sum of the series. Thus, too, if the fourth term, or ${}^{3}1=\frac{1}{4}$, is added to $\frac{1}{2}+\frac{1}{4}$, the first and second terms of the series, the aggregate will also be 1. Again, the fifth term, or ${}^{7}1=\frac{1}{8}$, added to $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$, will be equal to 1. And so in all other instances.

In like manner, in the series $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}$, &c. $=\frac{1}{2}$, *i. e.* $0+.1+{}^{5}1+{}^{17}1+{}^{53}1+{}^{161}1$, &c.,

if the third term $^{5}1$, or $\frac{1}{6}$ in our notation, is added to $\frac{1}{3}$, the first term; the sum will be $\frac{1}{2}$. And $^{17}1$, or $\frac{1}{18}$, the fourth term, added to the first and second terms of the series, viz. $\frac{1}{18}+\frac{1}{3}+\frac{1}{9}=\frac{9}{18}$ $=\frac{1}{2}$. Thus, too, $^{53}1$, or $\frac{1}{54}$, added to $\frac{1}{3}+\frac{1}{9}+$ $\frac{1}{27}=\frac{26}{54}=\frac{1}{2}$. And so in other instances.

Thus, too, in the series $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}$, &c., which, in our notation, is $0+{}^{2}1+{}^{11}1+{}^{47}1+{}^{191}1$, &c., it will be found that $\frac{1}{4}+{}^{11}1$, *i. e.* $\frac{1}{12}=\frac{1}{3}$, the sum of the series; that $\frac{1}{4}+\frac{1}{16}+\frac{1}{48}$, or $^{47}1=\frac{16}{48}$ $=\frac{1}{3}$; and that $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+{}^{191}1$, or $\frac{1}{192}=\frac{64}{192}=\frac{1}{3}$. And the like will take place in all other instances of this and other geometrical series.

30. In the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c., which in our notation, as we have before shown, will be $1+.1+{}^{2}1+{}^{3}1+{}^{4}1+{}^{5}1$, &c., and is produced by the expansion of the expression $\frac{1}{1-.1}=2$; if any term of the quotient is divided by the denominator of the expression, *i. e.* by $\frac{1}{2}$, the quotient will be equal to the sum of the remaining terms of the series, the term assumed being included. Thus, if the second term in our notation $.1$, or $\frac{1}{2}$, is divided by $\frac{1}{2}$, the quotient will be 1; which will be equal to the sum of $\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c. Again, if the third term $^{2}1$, or $\frac{1}{3}$, is divided by $\frac{1}{2}$,

the quotient $\frac{2}{3}$ will be evidently equal to $\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}$, &c., because $\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c. $=1$ and $\frac{2}{3}+\frac{1}{3}=1$. Thus, too, if the fourth term $^{3}1$, or $\frac{1}{4}$, is divided by $\frac{1}{2}$, the quotient will be $\frac{1}{2}$, and will be equal to the sum of $\frac{1}{10}+\frac{1}{15}+\frac{1}{21}$, &c. And this corroborates the truth of our theory. For, unless . 1, $^{2}1$, and $^{3}1$, were by their position, with reference to unity, equivalent to $\frac{1}{3}$, $\frac{1}{6}$, and $\frac{1}{10}$, the remaining terms of the series, in conjunction with those terms, would not be equal to what we have demonstrated them to be.

Again, in the series $1+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}$, &c. $=1\frac{1}{2}$, which, in our notation, will be $1+{}^{2}1+{}^{5}1+{}^{9}1={}^{14}1$, &c., if the second term $^{2}1$ or $\frac{1}{3}$ is assumed, and is divided by $\frac{2}{3}$, to which the denominator of the expression producing this series must be equivalent, the numerator being 1, then the quotient $\frac{1}{2}$ will be the sum of the remaining terms, viz. of the terms $\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}$, &c. Again, if the third term $^{5}1$ or $\frac{1}{6}$ is assumed, and is divided by $\frac{2}{3}$, the quotient $\frac{3}{12}$, or $\frac{1}{4}$, will be the aggregate of the remaining terms $\frac{1}{10}+\frac{1}{20}+\frac{1}{35}$, &c. Thus, too, if the fourth term $^{9}1$, or $\frac{1}{10}$, is divided by $\frac{2}{3}$, the quotient $\frac{3}{20}$ will be equal to

the sum of the remaining terms of the series: for $\frac{3}{20}+1+\frac{1}{4}+\frac{1}{10}=\frac{60}{40}=\frac{6}{4}=\frac{3}{2}$. And so in all other instances. Hence, though an expression cannot be found, the expansion of which gives this series, yet as the aggregate value of the series is $\frac{3}{2}$, the denominator of it must be equivalent to $\frac{2}{3}$, and the numerator to 1.

31. We have shown (in parag. 5), that in the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c., any series of terms, whose denominators differ from each other by 2, 3, 4, 5, 6, &c., are the $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, &c. parts of that series; and it will follow, from our method of notation, that as $1+.\,1+{}^{2}1+{}^{3}1+{}^{4}1+{}^{5}1$, &c.; *i. e.* as the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c., forms, by the position of its terms with reference to unity, the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\frac{1}{28}$, &c., so the half of it, viz. $0+1+{}^{3}1+{}^{5}1+{}^{7}1+{}^{9}1$, &c., *i. e.* $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}$, &c., will form, by position, the series $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}$, &c., which is the half of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c. And the other half of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, &c., viz. the series $1+{}^{2}1+{}^{4}1+{}^{6}1+{}^{8}1$, &c., or $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}$, &c., will form, by position, the series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}$, &c.; and we have before demon-

strated that the series $\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}$, &c., is produced by the expansion of an expression, the value of which is $\frac{1}{2}$.

Thus, too, the third of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, &c. will be, in our notation, $0+1+{}^{5}1+{}^{8}1+{}^{11}1$, &c.; *i. e.* will be $\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}$, &c., and this will form, by position, the series $\frac{1}{3}+\frac{1}{9}+\frac{1}{18}+\frac{1}{30}$, &c., which is $\frac{1}{3}$ of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c. But the series $1+\frac{1}{4}+\frac{1}{7}+\frac{1}{10}+\frac{1}{13}$, is also equal to $\frac{1}{3}$ of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, &c., as we have demonstrated (in parag. 5). And the terms of this series will form, by position with reference to unity, the series $1+\frac{1}{5}+\frac{1}{12}+\frac{1}{22}+\frac{1}{35}$, &c. As, likewise, the series $\frac{1}{9}+\frac{1}{18}+\frac{1}{30}+\frac{1}{45}$, &c. is commensurably $\frac{1}{3}$, so the series $\frac{1}{5}+\frac{1}{12}+\frac{1}{22}+\frac{1}{35}$, &c. is incommensurably $\frac{1}{3}$. For the series $1+\frac{1}{5}+\frac{1}{12}+\frac{1}{22}$, &c. is produced by the expansion of $\frac{1}{1-{}^{3}1}=\frac{1}{1-\frac{1}{4}}=\frac{4}{3}$. The like will also take place in the fourth, fifth, sixth, &c. parts, *ad infinitum*, of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, &c., when one of the series is that which arises from a multiplication of the denominator of each term of the above series by 4, 5, 6, &c., and the other series begins from unity.

Universally, also, of these parts of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, &c., the former will always be a commensurable, and the latter an incommensurable infinite series. And in a similar manner of the two series, which are formed by the position of the terms of the above series, with reference to unity, one will be commensurable, and the other incommensurable; viz. the series will be commensurable, which is produced by the position of the terms of the commensurable series, and the other will be incommensurable. Thus, for instance, the series $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}$, &c. is commensurable, but the series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}$, &c. is incommensurable.

32. As in decimals a number of infinite value, when it is placed below unity, becomes of a value less than unity; so likewise, in our method of notation, a fractional infinite series, the value of which is infinite, becomes, by its position with reference to unity, of a finite value. Thus, in decimals, 5000000, &c. *ad infin.* if placed below unity, becomes equal to $\frac{1}{2}$; and in our notation $.\,1+{}^{2}1+{}^{3}1+{}^{4}1+{}^{5}1$, &c., if unity is placed before it, will be $1+.\,1+{}^{2}1+{}^{3}1+{}^{4}1$, &c., and will be

equal to $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c. $\doteq 2$. Thus, too, $^{2}1+{}^{4}1+{}^{6}1+{}^{8}1$, &c. $=\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}$, &c., if unity is placed before it, will be $1+{}^{2}1+{}^{4}1+{}^{6}1$, &c. $=$ $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}$, &c. $=\frac{3}{2}$, incommensurably.

33. In our method of notation, the points correspond to the denominators, and the units annexed to them to the numerators of fractions. Hence, in the following instance, $1+.1+{}^{2}1+{}^{3}1$ $+{}^{4}1$, &c., or $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c., if the points in each term are doubled, another fractional series, the half of the former, will be produced; for, when the denominator of a simple fraction is doubled, the fraction is halved. But the above series will, in this case, become $1+{}^{2}1+{}^{4}1+{}^{6}1$ $+{}^{8}1$, &c., *i. e.* $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}$, &c.; and we have elsewhere shown that this series is incommensurably $1\frac{1}{2}$; or, in other words, is produced by an expression equivalent to $\frac{3}{2}$.

Thus, too, the series $1+{}^{3}1+{}^{6}1+{}^{9}1+{}^{12}1$, &c., *i. e.* $1+\frac{1}{5}+\frac{1}{12}+\frac{1}{22}+\frac{1}{35}$, &c., will be incommensurably $1\frac{1}{3}$, the points in each of the terms of the series being triple of each of those in the series $1+.1+{}^{2}1+{}^{3}1+{}^{4}1$, &c. And so in other instances.

As the series, likewise, $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}$, &c.

is the half of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c., and is incommensurable, so the series $0+{}^{2}1+{}^{4}1+{}^{6}1+{}^{8}1$, &c., or $0+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}$, &c., produced by the position of the terms $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}$, &c., with reference to unity, will also be incommensurable. Thus, too, $1+\frac{1}{4}+\frac{1}{7}+\frac{1}{10}+\frac{1}{13}$, &c., which is an incommensurable third part of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c., will, from the position of its terms $0+\frac{1}{4}+\frac{1}{7}+\frac{1}{10}$, &c., with reference to unity, produce the series $0+\frac{1}{5}+\frac{1}{12}+\frac{1}{22}$, &c., which will be an incommensurable third part of 1. And the like will take place in other instances.

34. In any simple fractional infinite series, where the denominators increase by a common multiplier, any term in our mode of notation, placed over a corresponding term of that series, will, when added to the preceding terms of the series, be equal to the sum of the series. But this will not be the case when the denominators are not produced by one and the same multiplier.

35. As the expression $\frac{1+.1}{1+1}$ is, when expanded, equal, according to our theory, to the series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}$, &c., and the expression $\frac{1}{1-..1}$,

when expanded, produces the series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}$, &c.; hence what is asserted by modern mathematicians is evidently true in these expressions. For they say that the latter of these series is $\frac{1}{6}$ of the square of the periphery of a circle whose diameter is 1. But the former of these series is equal to the area of such a circle, and four times the area is equal to the periphery. Four times, however, $\frac{1+.1}{1+1}=3$, and the square of 3 is 9. And $\frac{1}{1-..1}=\frac{3}{2}$, and six times $\frac{3}{2}$ is equal to 9.

36. In sect. 5, we have shown it to be universally true, that in the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, &c., if any series of terms, whose denominators are at an equal distance from each other, are subtracted from another series, the denominators of which are also at an equal distance from each other, and each of which differs from each of the denominators of the former series, by unity, the double of the series, which is the remainder of such subtraction, will always be a part of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c. And it is also remarkable that the distance of the denominators

of the terms of the parts of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$, &c., which are the remainders, when such parts are doubled, will always be at the same distance from each other as the denominators are of the terms of the series which are subtracted from each other. Thus, let the denominators of the terms to be subtracted be distant from each other by two intervening terms, then the denominators of the terms, when doubled of the series that is the remainder, will also be distant from each other by two intervening terms of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$, &c. For instance,

$$\begin{array}{l} 1+\frac{1}{4}+\frac{1}{7}+\frac{1}{10}+\frac{1}{13}+\frac{1}{16}, \text{ \&c.} \\ \underline{-\frac{1}{2}+\frac{1}{5}+\frac{1}{8}+\frac{1}{11}+\frac{1}{14}+\frac{1}{17}}, \text{ \&c.} \\ =\frac{1}{2}+\frac{1}{20}+\frac{1}{56}+\frac{1}{110}+\frac{1}{182}+\frac{1}{272}, \text{ \&c.} \end{array}$$

which, when doubled, will be

$$1+\frac{1}{10}+\frac{1}{28}+\frac{1}{55}+\frac{1}{91}+\frac{1}{136}, \text{ \&c.}$$

And the terms of this series are severally distant from each other by two terms of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c. Thus, too, in three intervals,

$$\begin{array}{l} 1+\frac{1}{5}+\frac{1}{9}+\frac{1}{13}+\frac{1}{17}, \text{ \&c.} \\ \underline{-\frac{1}{2}+\frac{1}{6}+\frac{1}{10}+\frac{1}{14}+\frac{1}{18}}, \text{ \&c.} \\ =\frac{1}{2}+\frac{1}{30}+\frac{1}{90}+\frac{1}{182}+\frac{1}{306}, \text{ \&c.} \end{array}$$

And this series doubled, is $1+\frac{1}{15}+\frac{1}{45}+\frac{1}{91}+\frac{1}{153}$, &c. The terms, also, of this series are distant

from the terms of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c., by three intervals. Thus, likewise, in four intervals,

$$
\begin{array}{l}
1+\frac{1}{6}+\frac{1}{11}+\frac{1}{16}+\frac{1}{21}, \text{ \&c.} \\
-\frac{1}{2}+\frac{1}{7}+\frac{1}{12}+\frac{1}{17}+\frac{1}{22}, \text{ \&c.} \\
\hline
=\frac{1}{2}+\frac{1}{42}+\frac{1}{132}+\frac{1}{272}+\frac{1}{462}, \text{ \&c.}
\end{array}
$$

The double of which is $1+\frac{1}{21}+\frac{1}{66}+\frac{1}{136}+\frac{1}{231}$, &c. And the like will take place in five, six, &c. intervals.

THE ELEMENTS

OF A

NEW ARITHMETICAL NOTATION,

ETC. ETC.

BOOK THE SECOND.

1. When any expression produces, by being expanded, an infinite series, I call that series a *distributed* value; which value is denoted by the expression, while it remains in the form through which it produces the series. Thus $\frac{1}{2-1}$ expanded is $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$, &c., which series is a *distributed* value, and is denoted by $\frac{1}{2-1}$; but the *undistributed* value is 1.

2. In infinite series, the value of which is finite, the *undistributed* value can be *actually*, or ἐν ἐνεργείᾳ assigned, as in the above instance; but in infinite series, the value of which is infinite, this is impos-

sible. For it can only be *potentially*, εν δυναμει, *virtually*, or *causally* assigned. Thus, in $\frac{1}{1-2+1}$ $=1+2+3+4+5+6$, &c., the *undistributed* value cannot be *actually* assigned.

3. In all infinite series, in which the *undistributed* is not equal to the *distributed* value, the former is incommensurable with the latter.

4. From what we have demonstrated, it appears that every infinite series of whole numbers has a twofold value; one, arising from the position of the terms, and the other being independent of position. Thus the infinite series $1+1+1+1+1$, &c., is, with reference to the position of its terms, $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c., but, when considered independently of position, it is an infinite series of units.

Thus, too, the series $1+2+3+4+5+6$, &c., is, from position, $1+\frac{2}{2}+\frac{3}{3}+\frac{4}{4}+\frac{5}{5}+\frac{6}{6}$, &c., *i. e.* is $1+1+1+1+1$, &c.; but without position, is the whole numbers $1+2+3+4$, &c. And so of the rest.

Hence, too, the square of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c., is the series $1+1+1+1+1+1$, &c., according to a *distributed* value. But, in

this case, this latter series is to be considered independently of position. This may be proved by a multiplication of the parts of this series into themselves: for, in such multiplication, the value of the product is nearly equal to the same number of the terms of the series $1+1+1+1+1$, &c. Thus $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{50}{24}$, and $\frac{50}{24}\times\frac{50}{24}=\frac{2500}{576}$ $=4\frac{1}{3}$ nearly. And four terms of the series $1+1+1+1+1$, &c., are equal to 4. Again, $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=\frac{137}{60}$, and $\frac{137}{60}\times\frac{137}{60}=\frac{18769}{3600}=5\frac{1}{5}$ nearly. And five terms of the series $1+1+1+1+1+1$, &c. $=5$. Thus, too, $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{882}{360}$, and $\frac{882}{360}\times\frac{882}{360}=\frac{777924}{129600}=6\frac{1}{400}$. And six terms of the series $1+1+1$, &c. $=6$. Again, $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}=\frac{363}{140}$, and this, squared, is $\frac{131769}{19600}$ $=6\frac{14169}{19600}$. And eight terms of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$, &c. $=\frac{761}{280}$, and this, squared, is $\frac{579121}{78400}=7\frac{30321}{78400}$. And so in other instances.

5. The following infinite series of whole numbers will be to each other according to their *distributed* form, inversely as their fractional reciprocals. In the first place, the infinite series of natural numbers $1+2+3+4+5+6$, &c., and the infinite series of odd numbers $1+3+5+7+9$

$+11$, &c., will, in their distributed form, be to each other as 1 to 2: for the expression, from the expansion of which the former is produced, is $\frac{1}{1-2+1}$, and that from which the latter is produced, is $\frac{1+1}{1-2+1}$. And in their reciprocals, $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c., $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}$, &c., the former will be to the latter as 2 to 1; as is evident from what we have demonstrated in parag. 4, Book I., and from what has also been demonstrated by modern mathematicians. In like manner the infinite series $1+4+7+10+13$, &c., produced by the expansion of $\frac{1+2}{1-2+1}$, will be, in its distributed form, to the series, $1+2+3+4+5+6$, &c.; *i. e.* to $\frac{1}{1-2+1}$, as 3 to 1. And in their reciprocals $1+\frac{1}{4}+\frac{1}{7}+\frac{1}{10}+\frac{1}{13}$, &c., $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c., the former will be to the latter as $\frac{1}{3}$ to 1. This, too, will be the case with the series $1+5+9+13+17$, &c., produced by the expansion of $\frac{1+3}{1-2+1}$, and the series $1+6+11+16+21$, &c., produced by the expansion of $\frac{1+4}{1-2+1}$, and their reciprocals $1+\frac{1}{5}+\frac{1}{9}+\frac{1}{13}+\frac{1}{17}$, &c., $1+\frac{1}{6}+\frac{1}{11}+\frac{1}{16}+\frac{1}{21}$, &c.; and, in short, with all the infinite parts of the series $1+\frac{1}{2}+\frac{1}{3}+$

$\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c., as is evident from what is demonstrated in the above-mentioned parag. 4.

The like also will take place in all polygonous series. Thus the series of squares $1+4+9+16+25+36$, &c., produced by the expansion of $\frac{1+1}{1-3+3-1}$, and the triangular series $1+3+6+10+15+21$, &c., produced from $\frac{1}{1-3+3-1}$, are obviously to each other as 2 to 1; and if unity is omitted in each, which is but an infinitesimal in each, their fractional reciprocals, which are $\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}$, &c., and $\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}$, &c., are, in connexion with unity, as we have shown in the first Book, produced by the expansion of $\frac{1}{1-.1}$ and $\frac{1}{1-..1}$, $=\frac{1}{1-\frac{1}{2}}$, $\frac{1}{1-\frac{1}{3}}=2$, and $1\frac{1}{2}$. And if unity is subtracted from each of these, the remainders will be 1 and $\frac{1}{2}$; and 1 is to $\frac{1}{2}$ as 2 to 1. Thus, too, the series of pentagons $1+5+12+22+35$, &c., produced by the expansion of $\frac{1+2}{1-3+3-1}$, will be to $\frac{1}{1-3+3-1}$, the expression from which the triangular series is produced, as 3 to 1. And with respect to their fractional reciprocals $1+\frac{1}{5}+\frac{1}{12}+\frac{1}{22}+\frac{1}{35}$, &c., is produced by the expansion of $\frac{1}{1-{}^{3}1}$: for this expression, as

we have shown in the first Book, is, when expanded, $1+{}^{3}1+{}^{6}1+{}^{9}1+{}^{12}1$, &c., and $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c. is produced from $\frac{1}{1-.1}$. But the former expression $\frac{1}{1-{}^{3}1}$, is $\frac{1}{1-\frac{1}{4}}=1\frac{1}{3}$, and the latter expression is $\frac{1}{1-.1}=\frac{1}{1-\frac{1}{2}}=2$. Hence, by subtracting unity from each, the latter series will be to the former as 3 to 1, *i.e.* as 1 to $\frac{1}{3}$; the pentagonal, as well as the tetragonal fractional series, being incommensurable. In like manner the series of hexagonal numbers $1+6+15+28+45$, &c., produced by the expansion of $\frac{1+3}{1-3+3-1}$, will be to $\frac{1}{1-3+3-1}$ as 4 to 1. And the fractional reciprocal of the former of these series, is produced by the expansion of $\frac{1}{1-{}^{4}1}$, and the latter by the expansion of $\frac{1}{1-.1}$. But the former expression $\frac{1}{1-{}^{4}1}$, is $\frac{1}{1-\frac{1}{5}}=1\frac{1}{4}$, and the latter is equal to 2. By subtracting, therefore, unity from each, the latter will be to the former as 4 to 1; the hexagonal as well as the pentagonal series being incommensurable. And the like will also take place with all the other polygonous

fractional series; all which, by our method, will be found to be incommensurable quantities.

The infinite series of heptagonal numbers $1+7+18+34$, &c., is produced by the expansion of the expression $\frac{1+4}{1-3+3-1}$; of octagons $1+8+21+40$, &c., by the expansion of $\frac{1+5}{1-3+3-1}$; of enneagons $1+9+24+46$, &c., by the expansion of $\frac{1+6}{1-3+3-1}$; and the expression $\frac{1+7}{1-3+3-1}$, when expanded, produces the infinite series of decagons $1+10+27+52$, &c. Hence, as $\frac{1}{1-3+3-1}$ is, when expanded, the infinite series of triangles, $\frac{1+1}{1-3+3-1}$ the infinite series of squares, $\frac{1+2}{1-3+3-1}$, the infinite series of pentagons, and $\frac{1+3}{1-3+3-1}$ the infinite series of hexagons, it follows that infinite series of polygonous numbers will be to each other in the ratio of the natural series of numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c. And, by our method, it may be demonstrated that this will also be true of their fractional reciprocals.

6. Geometrical infinite series, likewise, of whole numbers, will be to each other inversely as their

fractional reciprocals, according to their *distributed* form. Thus the series $1+2+4+8+16$, &c. will be to the series $1+3+9+27+81$, &c., inversely, as the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$, &c. is to the series $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}$, &c. For the expression, from the expansion of which the first of these series is produced, is $\frac{1}{1-2}$; that from which the second is produced is $\frac{1}{1-3}$; the third is an expansion of $\frac{1}{2-1}$; and the fourth of $\frac{1}{3-1}$. But $\frac{1}{1-2}$ is to $\frac{1}{1-3}$ as -1 to -2, *i. e.* as 1 to 2, and $\frac{1}{2-1}$ is to $\frac{1}{3-1}$ as 2 to 1. Thus, too, the series $1+4+16+64+256$, &c., will be to the series $1+5+25+125+625$, &c., in a distributed form, as 3 to 4; for the former is an expansion of the expression $\frac{1}{1-4}$; and the latter of $\frac{1}{1-5}$; and $1-4$ is to $1-5$ as -3 to -4, *i. e.* as 3 to 4. And, in their fractional reciprocals, the series $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}$, &c. will be to the series $\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\frac{1}{625}$, &c. as 4 to 3; for the former is produced by the expansion of $\frac{1}{4-1}$, and the latter by the expansion of $\frac{1}{5-1}$; and $\frac{1}{3}$ is to $\frac{1}{4}$ as 4 to 3. The

like analogy will also be found to take place in all other geometrical infinite series.

In the parts, also, of geometrical infinite series, the aggregates of whole numbers are inversely as the aggregates of their fractional reciprocals, as is evident from the following instances. The expression $\frac{1}{1+0-4}$, or $\frac{1}{1.-4}$, (see paragraph 6. Book I.), when expanded, gives the series $1+0+4+0+16+0+64$, &c. $=\frac{1}{3}$ of the series $1+2+4+8+16+32$, &c. For the latter series is produced by the expansion of $\frac{1}{1-2}$, and the denominator $1+0-4$, divided by $1-2$, gives for the quotient $1+2$, as may be seen by the operation below: —

$$\begin{array}{rl} 1-2) & 1+0-4 \quad (1+2 \\ & \underline{1-2} \\ & +2-4 \\ & \underline{+2-4} \\ & \end{array}$$

And the series $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}$, &c., is $\frac{1}{3}$ of the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$, &c. Again, the expression $\frac{1}{1+0+0-8}$ gives, when expanded, the series $1+0+0+8+0+0+64+0+0+512$, &c. $=\frac{1}{7}$ of the series $1+2+4+8+16+32$, &c. For the

denominator $1+0+0-8$, divided by the denominator $1-2$, gives for the quotient $1+2+4$.

$$\begin{array}{lll}
1-2) & 1+0+0-8 & (1+2+4 \\
 & \underline{1-2} & \\
 & \quad +2 & \\
 & \quad \underline{+2-4} & \\
 & \qquad\quad +4-8 & \\
 & \qquad\quad \underline{+4-8} & \\
 & \qquad\quad \;\cdot\;\; \cdot &
\end{array}$$

And the series $\frac{1}{8}+\frac{1}{64}+\frac{1}{512}+\frac{1}{4096}$, &c., is $\frac{1}{7}$ of the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$, &c.: for it is produced by the expansion of $\frac{1}{8-1}=\frac{1}{7}$. Again, as $\frac{1}{1+0-4}=1+0+4+0+16+0+64$, &c. is $\frac{1}{3}$ of the series $1+2+4+8+16+32$, &c., and as $\frac{0+2}{1+0-4}=0+2+0+8+0+32+0+128$, &c. is $\frac{2}{3}$ of the series $1+2+4+8+16$, &c., so $\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}$, &c. is $\frac{2}{3}$ of the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, &c.: for it is produced by the expansion of $\frac{1}{2-\frac{1}{2}}$, or of $\frac{2}{4-1}$. Farther still, $\frac{2}{1+0+0+0-16}=2+0+0+0+32+0+0+0+512$, &c., and is $\frac{2}{15}$ of the series $1+2+4+8+16$, &c., for $\frac{1+0+0+0-16}{1-2}=1+2+4+8$. And $\frac{8}{1+0+0+0-16}=8+0+0+0+128+0+0+0+2048$, &c., and is $\frac{8}{15}$ of the series $1+2+4+8+16+32$, &c. But $\frac{\frac{1}{2}}{1-\frac{1}{16}}=\frac{1}{2}$

$+\frac{1}{32}+\frac{1}{512}$, &c. $=\frac{8}{15}$. And $\frac{\frac{1}{8}}{1-\frac{1}{16}}=\frac{1}{8}+\frac{1}{128}+\frac{1}{2048}$, &c. And the series $\frac{1}{2}+\frac{1}{32}+\frac{1}{512}$, &c. is to the series $\frac{1}{8}+\frac{1}{128}+\frac{1}{2048}$, &c. as 8 to 2; and, therefore, is as the series $8+0+0+0+128+0+0+0+2048$, &c. to the series $2+0+0+0+32+0+0+0+512$, &c. Farther still, as $\frac{1}{1+0-9}=1+0+9+0+81+0+729$, &c. is $\frac{1}{4}$ of the series $1+3+9+27+81+243$, &c. (for $1+0-9$, divided by $1-3$, gives, for the quotient, $1+3$, and $\frac{1}{1-3}=1+3+9+27+81$, &c.); thus, also, $\frac{1}{9}+\frac{1}{81}+\frac{1}{729}$, &c. is $\frac{1}{4}$ of the series $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}$, &c., and is equal to $\frac{1}{8}$. For the series $\frac{1}{9}+\frac{1}{81}+\frac{1}{729}$, &c., is produced by the expansion of $\frac{1}{9-1}=\frac{1}{8}$*; and the series $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}$, &c. $=\frac{1}{2}$; and the $\frac{1}{4}$ of $\frac{1}{2}$ is equal to $\frac{1}{8}$. Also, as $\frac{1}{1+0+0-27}=1+0+0+27+0+0+729+0+0+19683$, &c.,

* This series is produced, by our method of notation, as follows: —

$$
\begin{array}{lll}
9-1) & 1 & (0+{}^{7}1+{}^{71}1+{}^{647}1, \text{ \&c.} \\
 & \underline{1-{}^{7}1} & \\
 & \quad +{}^{7}1 & \\
 & \quad \underline{+{}^{7}1-{}^{71}1} & \\
 & \qquad\quad +{}^{71}1 & \\
 & \qquad\quad \underline{+{}^{71}1-{}^{647}1} & \\
 & \qquad\qquad\quad +{}^{647}1, \text{ \&c.} &
\end{array}
$$

is $\frac{1}{13}$ of the series $1+3+9+27+81$, &c., (for $1+0+0+27$, divided by $1-3$, gives, for the quotient, $1+3+9$), so $\frac{1}{27}+\frac{1}{729}+\frac{1}{19683}$, &c. is $\frac{1}{13}$ of the series $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}$, &c. For this series is produced by the expansion of $\frac{1}{27-1}=\frac{1}{26}$; and the $\frac{1}{13}$ of $\frac{1}{2}$ is $\frac{1}{26}$. Again, $\frac{0+3}{1+0+0+0-81}=0+3+0+0+0+243+0+0+0+19683$, &c. $=\frac{3}{40}$ of the series $1+3+9+27+81$, &c., (for $\frac{1+0+0+0-81}{1-3}=1+3+9+27=40$). And $\frac{0+0+0+27}{1+0+0+0-81}=0+0+0+27+0+0+0+2187+0+0+0+177147$, &c. $=\frac{27}{40}$ of the series $1+3+9+27+81$, &c. Hence the former series is to the latter as 3 to 27. But $\frac{\frac{1}{3}}{1-\frac{1}{81}}=\frac{1}{3}+\frac{1}{243}+\frac{1}{19683}$, &c. $=\frac{81}{240}=\frac{27}{80}$. And $\frac{\frac{1}{27}}{1-\frac{1}{81}}=\frac{1}{27}+\frac{1}{2187}+\frac{1}{177147}$, &c. $=\frac{81}{2160}=\frac{3}{80}$.

7. Dodson, in vol. i. of his Mathematical Repository, has demonstrated that the infinite series of the fractional reciprocals of the figurate numbers $1+4+10+20+35$, &c. is equal to $\frac{3}{2}$, and also that the infinite series of the reciprocals of $1+5+15+35+70$, &c. is equal to $\frac{4}{3}$. But in a similar way it may be shown that the series $1+\frac{1}{6}+\frac{1}{21}+\frac{1}{56}+\frac{1}{126}$, &c. is equal to $\frac{5}{4}$. For 1+

$\frac{1}{5}+\frac{1}{15}+\frac{1}{35}+\frac{1}{70}$, &c. $=\frac{4}{3}$. Hence $\frac{4}{3}-\frac{1}{3}=\frac{1}{5}+\frac{1}{15}+\frac{1}{35}+\frac{1}{70}$, &c. And by subtraction $1=\frac{5-1}{5\,.\,1}+\frac{15-5}{15\,.\,5}+\frac{35-15}{35\,.\,15}+\frac{70-35}{70\,.\,35}$, &c. That is, $1=\frac{4}{5}+\frac{2}{15}+\frac{4}{105}+\frac{3\,5}{7\,0}$, &c. Hence, as $1+\frac{1}{4}=\frac{5}{4}$, if the series $\frac{4}{5}+\frac{2}{15}+\frac{4}{105}+\frac{3\,5}{7\,0}$, &c. is multiplied by $\frac{5}{4}$, the product will be $1+\frac{1}{6}+\frac{1}{21}+\frac{1}{56}$, &c. $=\frac{5}{4}$. Thus, too, it may be demonstrated that the series $1+\frac{1}{7}+\frac{1}{28}+\frac{1}{84}$, &c. is equal to $\frac{6}{5}$. And, in short, it will be found that the reciprocals of all the infinite series of figurate numbers beginning with the series $\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}$, &c., will form the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$, &c., *ad infinitum.*

The infinite series of figurate numbers, of which these series are the reciprocals, will be to each other as $1-1^1$, $1-1^2$, $1-1^3$, $1-1^4$, &c. For the series $1+3+6+10+15+21$, &c. is produced by the expansion of the expression $\frac{1}{1-3+3-1}$; but the series $1+4+10+20+35$, &c. by the expansion of $\frac{1}{1-4+6-4+1}$; and by dividing each of these expressions by $\frac{1}{1-2+1}$, the former will be to the latter as $1-1^1$ to $1-1^2$. Thus, too, the series $1+5+15+35+70$, &c. is produced by the expansion of the expression

$\frac{1}{1-5+10-10+5-1}$, and $\frac{1}{1-3+3-1}$, will be to this expression as $1-1^1$ to $1-1^3$. Again, the series $1+6+21+56$, &c. is produced by the expansion of the expression $\frac{1}{1-6+15-20+15-6+1}$, and $\frac{1}{1-3+3-1}$ will be to this expression as $1-1^1$ to $1-1^4$. And the like ratio will take place in all the other expressions from which infinite series of figurate numbers are produced. But as the series $\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c. is equal to 1, the series $\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}$, &c. $=\frac{1}{2}$, the series $\frac{1}{5}+\frac{1}{15}+\frac{1}{35}+\frac{1}{70}$, &c. $=\frac{1}{3}$, the series $\frac{1}{6}+\frac{1}{21}+\frac{1}{56}+\frac{1}{126}$, &c. $=\frac{1}{4}$; and so of the rest; thus, also, $1-1^2$ to 1 contains the ratio of $1-1$ to 1 twice, $1-1^3$ to 1 contains the ratio of $1-1$ to 1 thrice, $1-1^4$ contains the ratio of $1-1$ to 1 four times; and so of the rest. Hence, as 1 contains $\frac{1}{2}$ twice, so $1-1^2$ contains $1-1$ to 1 twice; and as 1 contains $\frac{1}{3}$ thrice, so $1-1^3$ contains $1-1$ to 1 thrice; and so of the rest. The aggregates, therefore, of the infinite series of figurate numbers are to each other, with respect to number of ratios, what the aggregates of their reciprocals are to each other with respect to numerical quantity; unity being excepted in each of the

figurate series, because, as they are whole numbers, it is but an infinitesimal in the value of the series.

8. As the value of the expression, from the expansion of which the series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}$, &c. is produced, is less than the sum arising from the addition of the parts of this series, this also would be the case with the expressions, if they could be found, from which the series $1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\frac{1}{125}$, &c., and the series $1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\frac{1}{625}$, &c. are evolved; and in a similar manner with the series of all the other powers of the terms of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, &c. But this is indicated by the expressions from which the whole numbers, of which these fractional series are the reciprocals, are evolved. For the numerators of the expressions $\frac{1+1}{1-3+3-1}$, $\frac{1+4+1}{1-4+6-4+1}$, $\frac{1+11+11+1}{1-5+10-10+5-1}$, &c. are in a distributed form, which universally indicates that the value of the expressions, from which the fractional reciprocals are expanded, is less than the sums arising from the addition of the parts of the series.

9. Hence there are many infinite fractional

series, the aggregate value of which is finite, that are similar to infinite series, the parts of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c., and which have an infinite aggregate value. For the aggregate of $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}$, &c. is half the aggregate of the whole series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, &c., and therefore the aggregate of $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}$, &c. is also the half of the whole series; and yet every term in the former is greater than every corresponding term in the latter series; viz. the first term of the former is greater than the first term of the latter; the second of the former, than the second of the latter; and so on. This will also be the case with the series $1+\frac{1}{4}+\frac{1}{7}+\frac{1}{10}+\frac{1}{13}+\frac{1}{16}$, &c., and the series $\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}$, &c., each of which is $\frac{1}{3}$ of the whole series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c., though every term in the former is greater than every corresponding term in the latter series. The like will take place in all the parts of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, &c.

10. It is shown by Johan. Bernouli, (Op. tom. iv. p. 11, &c.), that if the denominators of infinite series of simple fractions are any power of the natural numbers, the sum of the odd will be to the sum of the even terms, as the same

power diminished by unity is to unity. Thus, in the series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}$, &c., the sum of the odd is to the sum of the even terms as 3 to 1. Thus, too, in the series $1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\frac{1}{125}+\frac{1}{216}$, &c., the ratio of the terms is that of 7 to 1. And in the series $1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\frac{1}{625}$, &c., the ratio is that of 15 to 1.

This will be found to be indicated by the series, of which the above series are the reciprocals, as follows: $\frac{1+1}{1-3+3-1}=1+4+9+16+25+36$, &c., and $\frac{1+6+1}{1-3+3-1}=1+9+25+49+81$, &c. Subtract the numerator $1+1$ from the numerator $1+6+1$, and the remainder is 6. And the ratio of 6 to 2, *i. e.* the ratio of 3 to 1, will be the ratio of the sum of all the odd fractional terms to the sum of all the even terms; viz. $1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}$, &c. will be to $\frac{1}{4}+\frac{1}{16}+\frac{1}{36}$, &c. as 3 to 1.

Again, $\frac{1+4+1}{1-4+6-4+1}=1+8+27+64+125+216$, &c., $\frac{1+23+23+1}{1-4+6-4+1}=1+27+125+343+729$, &c. Subtract, therefore, the numerator $1+4+1$ from the numerator $1+23+23+1$, *i. e.* 6 from 48, and the remainder is 42. And the ratio of 42 to 6 is that of 7 to 1, and is the ratio of

the sum of all the odd fractional terms to the sum of all the even terms; viz. $1+\frac{1}{27}+\frac{1}{125}+\frac{1}{343}$, &c. will be to $\frac{1}{8}+\frac{1}{64}+\frac{1}{216}$, &c. as 7 to 1.

Thus, too, $\frac{1+11+11+1}{1-5+10-10+5-1} = 1+16+81+256+625=1296$, &c., and $\frac{1+76+230+76+1}{1-5+10-10+5-1} = 1+81+625+2401$, &c. Let the less numerator, therefore, be subtracted from the greater, and the remainder will be 360. And the ratio of 360 to 24, the less numerator, is that of 15 to 1; which is the ratio of the sum of all the odd fractional terms $1+\frac{1}{81}+\frac{1}{625}+\frac{1}{2401}$, &c. to the sum of all the even terms $\frac{1}{16}+\frac{1}{256}+\frac{1}{1296}$, &c. And so in all other instances.

11. In every series of terms in arithmetical or geometrical progression, or in any progression in which the terms mutually exceed each other, the last term is equal to the first term added to the second term, diminished by the first; added to the third term, diminished by the second; added to the fourth term, diminished by the third; and so on. And if the number of terms be infinite, the last term is equal to the series multiplied by $1-1$.

Let the terms, whatever the series may be,

be represented by a, b, c, d, e, then $a+b-a+c-b+d-c+e-d=e$.

$$\begin{array}{l} \text{For } a \\ +b-a \\ +c-b \\ +d-c \\ +e-d \\ \hline =e. \end{array}$$

But if the number of terms be infinite, viz. if the series be $a+b+c+d+e+f+g$, &c. *ad infin.*, then this series, multiplied by $1-1$, will be

$$\begin{array}{l} a+b+c+d+e+f+g, \text{ \&c.} \\ 1-1 \\ \hline a+b+c+d+e+f+g, \text{ \&c.} \\ \quad -a-b-c-d-e-f, \text{ \&c.} \\ \hline =a+b-a+c-b+d-c+e-d+f-e+g-f, \text{ \&c.} \end{array}$$

Hence, as this proposition equally applies to fractional as well as to integral infinite series, the last terms of a great variety of fractional infinite series may be obtained; and in each of these, the last term, multiplied by the number of terms, will be equal to the sum of the series. Thus, $\frac{1-1}{2-1}=\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\frac{1}{16}-\frac{1}{32}$, &c. is the last term of the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$, &c. $=\frac{1}{2-1}$; and $\frac{1-1}{2-1}\times\frac{1}{1-1}=\frac{1}{2-1}$. Thus, too, in the series $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}$, &c. $=\frac{1}{3-1}$, the last term is $\frac{1-1}{3-1}$,

and $\frac{1-1}{3-1} \times \frac{1}{1-1} = \frac{1}{3-1}$. And in a similar manner in all geometrical fractional series.

Thus, also, in the series $\frac{1}{1}+\frac{3}{2}+\frac{7}{4}+\frac{15}{8}+\frac{31}{16}+\frac{63}{32}$, &c., (see Bonnycastle's Algebra, p. 168,) formed by the successive sums of the geometrical progression $\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, &c., the aggregate of this geometrical series will be the last term of $\frac{1}{1}+\frac{3}{2}+\frac{7}{4}$, &c. But $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$, &c. is produced by the expansion of $\frac{2}{2-1}$, and this multiplied by $\frac{1}{1-1}$, the number of the terms will be $\frac{2}{2-3+1} = \frac{1}{1}+\frac{3}{2}+\frac{7}{4}+\frac{15}{8}$, &c. Again, the series $1+\frac{4}{3}+\frac{13}{9}+\frac{40}{27}$, &c. is formed by the successive sums of the geometrical progression $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}$, &c., and the aggregate of this geometrical series will be the last term of $1+\frac{4}{3}+\frac{13}{9}+\frac{40}{27}$, &c. But $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}$, &c. is produced by the expansion of $\frac{3}{3-1}$; and this, multiplied by the number of terms $\frac{1}{1-1}$, will be the sum of the series in a *distributed* form, viz. will be $\frac{3}{3-4+1} = 1+\frac{4}{3}+\frac{13}{9}+\frac{40}{27}$, &c. Thus, likewise, in the series $1+\frac{5}{4}+\frac{21}{16}+\frac{85}{64}+\frac{341}{256}$, &c., which is formed by the addition of the terms of the series $1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}$, &c., the last term of it will be $\frac{4}{4-1}$; which,

when expanded, is the series $1+\frac{1}{4}+\frac{1}{16}$, &c.; and this, multiplied by the number of terms $\frac{1}{1-1}$, will be the sum of the series, *i. e.* will be $\frac{4}{4-5+1}$ $=1+\frac{5}{4}+\frac{21}{16}+\frac{85}{64}$, &c. And the like will take place in all other series produced by the addition of the terms of other similar geometrical progressions.

In these series, the difference arising from the multiplication of the number of terms $\frac{1}{1-1}$ by the expressions $\frac{2}{2-1}$, $\frac{3}{3-1}$, $\frac{4}{4-1}$, and the multiplication of the said number of terms by the above expressions, in an *undistributed* form, is remarkable. Thus, for instance, $\frac{1}{1-1}\times\frac{2}{2-1}$ $=\frac{2}{2-3+1}$; and this expanded is the series $\frac{1}{1}+\frac{3}{2}$ $+\frac{7}{4}+\frac{15}{8}$, &c. But $\frac{1}{1-1}\times\frac{2}{1}=\frac{2}{1-1}$, and this expanded is the infinite series $2+2+2+2+2$, &c.; and the difference between the two is the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, &c. $=2$. Again, $\frac{3}{3-1}$ $\times\frac{1}{1-1}=\frac{3}{3-4+1}$; and this, when expanded, is $1+\frac{4}{3}+\frac{13}{9}+\frac{40}{27}$, &c. But $\frac{3}{2}\times\frac{1}{1-1}=\frac{3}{2-2}=$ $\frac{3}{2}+\frac{6}{4}+\frac{12}{8}+\frac{24}{16}$, &c. $=\frac{3}{2}+\frac{3}{2}+\frac{3}{2}+\frac{3}{2}$, &c., and the

difference between the two is the series $\frac{1}{2}+\frac{1}{6}+\frac{1}{18}+\frac{1}{54}$, &c. $=\frac{3}{5}=\frac{1}{2-\frac{1}{3}}$. Thus, too, $\frac{4}{4-1}\times\frac{1}{1-1}=\frac{4}{4-5+1}$; and this, when expanded, is $\frac{1}{1}+\frac{5}{4}+\frac{21}{16}+\frac{85}{64}+\frac{341}{256}$, &c. But $\frac{1}{1-1}\times\frac{4}{3}=\frac{4}{3-3}=\frac{4}{3}+\frac{12}{9}+\frac{36}{27}$, &c. $=\frac{4}{3}+\frac{4}{3}+\frac{4}{3}+\frac{4}{3}$, &c.; and the difference between the two series is $\frac{1}{3}+\frac{1}{12}+\frac{1}{48}+\frac{1}{192}$, &c., produced by the expansion of $\frac{1}{3-\frac{3}{4}}=\frac{4}{9}$.

Hence, likewise, in every infinite series, whether fractional or integral, the terms of which have an uninterrupted continuity, the last term, multiplied by the number of terms, will be equal to the sum of the series. For the last term of every infinite series is equal to the fraction by which that series is produced, multiplied by $1-1$; and as the number of terms is $\frac{1}{1-1}$, it is evident that this last term, multiplied by $\frac{1}{1-1}$, will be equal to the sum of the series. It follows, therefore, that all infinite series, in a *distributed* form, are to each other as their last terms, viz. as the terms which are obtained by multiplying those series by $1-1$.

12. The following propositions, therefore, in Dr. Wallis's Arithmetic of Infinites, are not true

of the infinite series of whole numbers, which he there adduces, according to their *distributed*, but according to their *undistributed** value, or the value obtained from the aggregation of the terms by induction, viz.:

"If a series of numbers in arithmetical progression begin with a cipher, and the common difference be 1, if the last term be multiplied into the number of terms, the product will be double the sum of all the series.

"If a series of squares, whose sides or roots are in arithmetical progression beginning with a cipher, be infinitely continued, the last term being multiplied into the number of terms, will be triple the sum of all the series.

"If a series of cubes, whose roots are in arithmetical progression, beginning with a cipher, be infinitely continued, the last term multiplied into the number of terms, will be quadruple the sum of all the series.

"If a series of biquadrats, whose roots are

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* The difference between the *distributed* and *undistributed* value of infinite series, was not fully perceived by me when I wrote my Elements of the True Arithmetic of Infinites.

in arithmetical progression, beginning with a cipher, be infinitely continued, the last term multiplied into the number of terms, will be quintuple the sum of all the series.

For the number of terms in each of these series must be $0+1+1+1+1+1$, *ad infinitum*, or $\frac{0+1}{1-1}$, as there cannot, in these series, be a greater number of terms than this; and the continuity in each of these series is uninterrupted. In the first of these series, therefore, the last term will be $\frac{0+1}{1-1} = 0+1+1+1+1$, &c., and $\frac{0+1}{1-1} \times \frac{0+1}{1-1}$, the number of the terms will be equal to $\frac{0+1}{1-2+1} = 0+1+2+3+4+5+6$, &c. In the second, the last term will be $\frac{0+1+1}{1-2+1} = 0+1+3+5+7+9+11$, &c., and $\frac{0+1+1}{1-2+1} \times \frac{0+1}{1-1}$, the number of the terms will be equal to $\frac{0+1+1}{1-3+3-1} = 0+1+4+9+16+25+36$, &c. The last term in the third will be $\frac{0+1+4+1}{1-3+3-1} = 0+1+7+19+37+61$, &c., and $\frac{0+1+4+1}{1-3+3-1} \times \frac{0+1}{1-1}$, the number of the terms will be equal to $\frac{0+1+4+1}{1-4+6-4+1} = 0+1+8+27+64+125$, &c. And in the fourth the last

term will be $\frac{0+1+11+11+1}{1-4+6-4+1}$, and $\frac{0+1+11+11+1}{1-4+6-4+1}$ $\times \frac{0+1}{1-1}$ the number of the terms, will be $\frac{0+1+11+11+1}{1-5+10-10+5-1}$.

13. As when any finite number of terms of these infinite series is taken, the last term multiplied by the number of terms will by no means be equal to the sum of the series, but when the series begin from unity, will continually diverge from such equality, as will be found upon trial to be the case; hence, in infinite series, when they are in a *distributed* form, we cannot always reason from the aggregate of the parts to the aggregate of the whole, or from the aggregate of the whole to the aggregate of the parts.

Dr. Cheyne, not having perceived this truth, asserts, in his Philosophical Principles of Religion, p. 148, that the series $1+2+3+4+5+6$, &c., which is an expansion of $\frac{1}{1-2+1}$, is but half the square of $1+1+1+1+1$, &c. the expansion of $\frac{1}{1-1}$, though $\frac{1}{1-2+1}$ is the square of $\frac{1}{1-1}$. It might be easy to confute this assertion by merely observing, that if $\frac{1}{1-2+1}$ is the square of

$\frac{1}{1-1}$, the series $1+2+3+4+5+6$, &c. must be the square of the series $1+1+1+1+1$, &c.; for, if it is not, the algebraic rules of subtraction and division are false, and also the common mode of multiplication; and it is strange that any man in his senses, who understood the elements of arithmetic and algebra, should make such an assertion. But, lest some one should still pervicaciously contend that Dr. Cheyne is right, the following is a complete demonstration of the contrary.

To multiply $1+1+1+1+1$, &c. by itself, is the same thing, by the fifteenth definition of the seventh Book of Euclid, as to add this series to itself $1+1+1+1+1$, &c. times, viz. it will be equal to $\frac{1}{1-1}+\frac{1}{1-1}+\frac{1}{1-1}+\frac{1}{1-1}$, &c. infinitely, or to $\frac{1+1+1+1+1+1, \text{\&c.}}{1-1} = 1+2+3+4+5+6$, &c. This, however, is owing to the wonderful nature of the infinite. That the infinite, indeed, has a power very different from the finite, was not unknown, in some instances, to Dr. Wallis, as is evident from the following observation made by him in his Arithmetica Infinitorum, pp. 131, 132. "Si series subsecundanorum ali-

quousque continuetur, putà $\sqrt{0}+\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{5}+\sqrt{6}$, ipsius ratio ad maximum toties positum, putà $7\sqrt{6}$, non videtur aliàs explicabilis quàm $\frac{\sqrt{0}+\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{5}+\sqrt{6}}{7\sqrt{6}}$ vel $\frac{0+1+\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{5}+\sqrt{6}}{7\sqrt{6}}$. Verùm si eadem series supponatur in infinitum continuanda, prodibit tandem ratio $\frac{2}{3}$, vel 2 ad 3, aut 1 ad $1\frac{1}{2}$, ut dictum est prop. 53. 54. ipsâ quidem infinitate (*quod mirum videatur*) irrationabilitatem destruente."

14. If any infinite series of whole numbers, which increase either in an arithmetical or a geometrical ratio, is divided by $1+1$, then the sum of the first two terms of the quotient will be equal to the second term of the series; of the second two, to the fourth term; of the third two, to the sixth term; of the fourth two, to the eighth term, and so on *ad infinitum.*

Thus the quotient of the series $1+2+3+4+5+6+7$, &c. divided by $1+1$, is $1+1+2+2+3+3+4+4+5+5+6+6$, &c. And the sums of the couples of terms are $2+4+6+8+10$, &c.

Thus, too, the quotient of the series $1+3+6+10+15+21+28$, &c., divided by $1+1$, is $1+2+$

$4+6+9+12+16$, &c. And the sums of the couples of terms are $3+10+21$, &c.

In like manner the quotient of the series $1+2+4+8+16+32$, &c., divided by $1+1$, is $1+1+3+5+11+21$, &c. And the sums of the couples of terms are $2+8+32$, &c.

But if such series are divided by $1+1+1$, then the sum of the first three terms will be equal to the third term of the series; of the second three, to the sixth term; of the third three, to the ninth term, and so on.

Thus, in the series $1+2+3+4+5+6+7+8$, &c., the quotient resulting from the division of it by $1+1+1$, is $1+1+1+2+2+2+3+3+3+4+4+4$, &c., viz. is equal to $3+6+9+12$, &c. And in the series $1+2+4+8+16+32$, &c. the quotient is $1+1+2+5+9+18+37$, &c. which is equal to $4+32+256$, &c.

In a division by $1+1+1+1$, the like will take place in the sums of every four terms of the quotient.

Thus, in the series $1+2+3+4+5+6$, &c. the quotient will be $1+1+1+1+2+2+2+2+3+3+3+3+4+4+4+4$, &c. and this will be equal to

$4+8+12+16$, &c. And in the series $1+2+4+8+16+32$, &c. the quotient is $1+1+2+4+9+18+34+67$, &c. which is equal to $8+128$, &c.

In a division, also, by 5, 6, 7, &c., unities in this distributed form, the like will take place in the sums of 5, 6, 7, &c. terms of the quotient.

This will likewise be the case in infinite series of fractional terms.

Thus, if the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$, &c. $=1$, is divided by $1+1$, the quotient will be $\frac{1}{2}-\frac{1}{4}+\frac{3}{8}-\frac{5}{16}+\frac{11}{32}-\frac{21}{64}$, &c., equal, when the terms are taken separately, to $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}$, &c. $=\frac{1}{3}$, instead of $\frac{1}{2}$.

If, also, the series $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}$, &c. $=\frac{1}{2}$, is divided by $1+1$, the quotient will be $\frac{1}{3}-\frac{2}{9}+\frac{5}{27}-\frac{14}{81}+\frac{41}{243}-\frac{122}{729}$, &c. And if the terms of the quotient are taken separately, they will be $\frac{1}{9}+\frac{1}{81}+\frac{1}{729}$, &c. $=\frac{1}{8}$, instead of $\frac{1}{4}$. Thus, too, if the series $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\frac{1}{1024}$, &c. $=\frac{1}{3}$, is divided by $1+1$, the quotient will be $\frac{1}{4}-\frac{3}{16}+\frac{13}{64}-\frac{51}{256}+\frac{205}{1024}-\frac{819}{4096}$, &c. And the terms of the quotient, taken separately, will be $\frac{1}{16}+\frac{1}{256}+\frac{1}{4096}$, &c. $+\frac{1}{15}$, instead of $\frac{1}{6}$: for $\frac{1}{3}$ divided by $2=\frac{1}{6}$.

The following instances also clearly demonstrate the difference between the *distributed* and *undistributed* value of infinite fractional series. $\frac{2+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}}{1+1}$, &c. $= 2-1+\frac{3}{2}+\frac{5}{4}+\frac{11}{8}-\frac{21}{16}+\frac{43}{32}-\frac{85}{64}$, &c. $=$ if the terms are taken separately $1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}$, &c. $= 1\frac{1}{3}$, instead of 2. $\frac{4+2+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}}{1+1}$, &c. $= 4-2+3-\frac{5}{2}+\frac{11}{4}-\frac{21}{8}+\frac{43}{16}-\frac{85}{32}$, &c. $=$ when the terms are taken separately $2+\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}$, &c. $= \frac{8}{3}$, instead of 4. $\frac{8+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}}{1+1}$, &c. $= 8-\frac{15}{2}+\frac{31}{4}-\frac{61}{8}+\frac{123}{16}-\frac{245}{32}+\frac{491}{64}-\frac{981}{128}$, &c. $= \frac{2}{3}$, instead of $4\frac{1}{2}$. And this quotient $8-\frac{15}{2}+\frac{31}{4}-\frac{61}{8}+\frac{123}{16}-\frac{245}{32}$, &c., divided by $2-1$, will be $4-\frac{7}{4}+\frac{24}{8}-\frac{37}{16}+\frac{86}{32}-\frac{159}{64}+\frac{332}{128}-\frac{649}{256}$, &c. $=$ when the terms are taken separately $\frac{9}{4}+\frac{11}{16}+\frac{13}{64}+\frac{15}{256}$, &c. But this last series is equal to $\frac{9+\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}}{4-1}$, &c. $= \frac{29}{9}$, or $3\frac{2}{9}$ instead of $\frac{2}{3}$, when the terms are separately taken.

Again, if $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$, &c. be divided by $\frac{1}{2}+\frac{1}{2}$, the quotient will be $1-\frac{1}{2}+\frac{6}{8}-\frac{10}{16}+\frac{22}{32}-\frac{42}{64}$, &c. $=$ when the terms are taken separately $\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}$, &c. $= \frac{2}{3}$, instead of 1. And if the same dividend be divided by $\frac{1}{4}+\frac{1}{4}$, the quotient will be $2-1+\frac{12}{18}-\frac{20}{16}+\frac{44}{32}-\frac{84}{64}$, &c. $= 1+\frac{1}{4}$

$+\frac{1}{16}+\frac{1}{64}$, &c. $= 1\frac{1}{3}$, instead of 1. If, also, the same dividend be divided by $\frac{1}{8}+\frac{1}{8}$, the quotient, when the terms are taken separately, will be equal to $\frac{8}{3}$; if by $\frac{1}{16}+\frac{1}{16}$, the quotient will be equal to $\frac{16}{3}$; if by $\frac{1}{32}+\frac{1}{32}$, the quotient will be equal to $\frac{32}{3}$; and if by $\frac{1}{64}+\frac{1}{64}$, the quotient will be $\frac{64}{3}$.

Hence, in the first instance, the quotient is less than the *undistributed* value by $\frac{1}{3}$. In the second, the deficiency is equal to $\frac{2}{3}$. In the third, to $\frac{4}{3}$. In the fourth, to $\frac{8}{3}$. In the fifth, to $\frac{16}{3}$. And in the sixth, to $\frac{32}{3}$.

If the series $100+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$, &c., be divided by $1+1$, the quotient will be $100 - \frac{199}{2} + \frac{399}{4} - \frac{797}{8} + \frac{1595}{16} - \frac{3189}{32} + \frac{6379}{64} - \frac{12757}{128}$, &c. Let this quotient be divided by $2-1$, as in the next page.

$$2-1)\ 100-\tfrac{199}{2}+\tfrac{399}{4}-\tfrac{797}{8}+\tfrac{1595}{16}-\tfrac{3189}{32}+\tfrac{6379}{64}-\tfrac{12757}{128}, \text{ \&c.}\quad (50-\tfrac{99}{4}+\tfrac{300}{8}-\tfrac{497}{16}+\tfrac{1098}{32}-\tfrac{2091}{64}+\tfrac{4288}{128}-\tfrac{8469}{256}, \text{ \&c.}$$

$$\begin{array}{l}
100-50 \\
\hline
-\tfrac{99}{2}+\tfrac{399}{4} \\
-\tfrac{99}{2}+\tfrac{99}{4} \\
\hline
\quad +\tfrac{300}{4}-\tfrac{797}{8} \\
\quad +\tfrac{300}{4}-\tfrac{300}{8} \\
\hline
\quad -\tfrac{497}{8}+\tfrac{1595}{16} \\
\quad -\tfrac{497}{8}+\tfrac{497}{16} \\
\hline
\qquad +\tfrac{1098}{16}-\tfrac{3189}{32} \\
\qquad +\tfrac{1098}{16}-\tfrac{1098}{32} \\
\hline
\qquad\quad -\tfrac{2091}{32}+\tfrac{6379}{64} \\
\qquad\quad -\tfrac{2091}{32}+\tfrac{2091}{64} \\
\hline
\qquad\qquad +\tfrac{4288}{64}-\tfrac{12757}{128} \\
\qquad\qquad +\tfrac{4288}{64}-\tfrac{4288}{128} \\
\hline
\qquad\qquad\quad -\tfrac{8469}{128}, \text{ \&c.}
\end{array}$$

$$=\tfrac{101}{4}+\tfrac{103}{16}+\tfrac{105}{64}+\tfrac{107}{256}, \text{ \&c.}$$

$$=\frac{101+\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}, \text{ \&c.}}{4-1}$$

$$=\frac{101+\frac{2}{3}}{4-1}=\tfrac{305}{9}=33\tfrac{8}{9}.$$

But the value of the series $100-\frac{199}{2}+\frac{399}{4}-\frac{797}{8}+\frac{1595}{16}-\frac{3189}{32}$, &c., when the terms are taken separately, is no more than $\frac{2}{3}$. For $100-\frac{199}{2}$ is equal to $\frac{1}{2}$, $\frac{399}{4}-\frac{797}{8}$ is equal to $\frac{4}{32}=\frac{1}{8}$, $\frac{1595}{16}-\frac{3189}{32}$ is equal to $\frac{1}{32}$, and $\frac{6379}{64}-\frac{12757}{128}$ is equal to $\frac{1}{128}$, and so of the rest of the terms. And the series $\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}$, &c., is equal to $\frac{2}{3}$. So that, in the above instance, the value of the quotient is $33\frac{8}{9}$, instead of $\frac{2}{3}$.

The following instances also are confirmations of the difference between *distributed* and *undistributed* value:

1+1) $\frac{3}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$, &c. $(\frac{3}{4}-\frac{5}{8}+\frac{11}{16}-\frac{21}{32}+\frac{43}{64}-\frac{85}{128}$, &c. $=\frac{1}{8}+\frac{1}{32}+\frac{1}{128}$, &c. $=\frac{1}{6}$, instead of $\frac{1}{2}$.

1+1) $\frac{7}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}$, &c. $(\frac{7}{8}-\frac{13}{16}+\frac{27}{32}-\frac{53}{64}+\frac{107}{128}-\frac{213}{256}$, &c. $=\frac{1}{16}+\frac{1}{64}+\frac{1}{256}$, &c. $=\frac{1}{12}$, instead of $\frac{1}{2}$.

$$
\begin{array}{l}
\frac{7}{8}+\frac{7}{8} \\
\hline
-\frac{13}{16}+\frac{1}{32} \\
-\frac{13}{16}-\frac{13}{16} \\
\hline
\quad +\frac{27}{32}+\frac{1}{64} \\
\quad +\frac{27}{32}+\frac{27}{32} \\
\hline
\qquad -\frac{53}{64}+\frac{1}{128} \\
\qquad -\frac{53}{64}-\frac{53}{64} \\
\hline
\qquad\quad +\frac{107}{128}+\frac{1}{256} \\
\qquad\quad +\frac{107}{128}+\frac{107}{128} \\
\hline
\qquad\qquad -\frac{213}{256}, \text{ \&c.}
\end{array}
$$

1+1) $\frac{15}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\frac{1}{512}$, &c. $(\frac{15}{16}-\frac{29}{32}+\frac{59}{64}-\frac{117}{128}+\frac{235}{256}-\frac{469}{512}$, &c. $=\frac{1}{32}+\frac{1}{128}+\frac{1}{512}$, &c. $=\frac{1}{24}$, instead of $\frac{1}{2}$.

$$
\begin{array}{l}
\frac{15}{16}+\frac{15}{16} \\
\hline
-\frac{29}{32}+\frac{1}{64} \\
-\frac{29}{32}-\frac{29}{32} \\
\hline
\quad +\frac{59}{64}+\frac{1}{128} \\
\quad +\frac{59}{64}+\frac{59}{64} \\
\hline
\qquad -\frac{117}{128}+\frac{1}{256} \\
\qquad -\frac{117}{128}-\frac{117}{128} \\
\hline
\qquad\quad +\frac{235}{256}+\frac{1}{512} \\
\qquad\quad +\frac{235}{256}+\frac{235}{256} \\
\hline
\qquad\qquad -\frac{469}{512}, \text{ \&c.}
\end{array}
$$

And if $\frac{31}{32}+\frac{1}{64}+\frac{1}{128}$, &c. be divided by $1+1$, the quotient will be equal to $\frac{1}{48}$ instead of $\frac{1}{2}$. By which it appears that the quotients in such series continually decrease in a subduple ratio; the quotient in the second instance being one half of the quotient in the first instance; the quotient in the third being one half of that in the second instance; and so on *ad infinitum*, when the terms of the quotients are collected separately. Hence it follows, that when all the terms of the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, &c. are collected into one sum, or, in other words, into the expression $\frac{1}{2-1}$, the quotient arising from this expression, divided by $1+1$, will be equal to 0, when the terms of the quotient are taken separately; and this is evident from what follows:

$$
\begin{array}{l}
1-1\Big)\ \dfrac{1}{2-1}\ \Big(\dfrac{1}{2-1}-\dfrac{1}{2-1}+\dfrac{1}{2-1}-\dfrac{1}{2-1},\text{ \&c.}\\
\qquad\quad \underline{\dfrac{1}{2-1}+\dfrac{1}{2-1}}\\
\qquad\qquad\qquad -\dfrac{1}{2-1}\\
\qquad\qquad\quad \underline{-\dfrac{1}{2-1}-\dfrac{1}{2-1}}\\
\qquad\qquad\qquad\qquad +\dfrac{1}{2-1},\text{ \&c.}
\end{array}
$$

$1+1)\ 2+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$, &c. $(2-\frac{3}{2}+\frac{7}{4}-\frac{13}{8}+\frac{27}{16}-\frac{53}{32}$, &c. $=\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}$, &c. $=\frac{2}{3}$, instead of $1\frac{1}{2}$.

$1+1)\ \ 3+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$, &c. $(3-\frac{5}{2}+\frac{11}{4}-\frac{21}{8}+\frac{43}{16}-\frac{85}{32}$, &c. $=\frac{1}{2}+\frac{1}{8}+\frac{1}{32}$, &c., $=\frac{2}{3}$, instead of 2.

In like manner, the quotient of $4+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, &c. divided by $1+1$, will be $\frac{2}{3}$, instead of $2\frac{1}{2}$. And this will be the case whatever whole number is added to the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, &c. when the series, with such addition, is divided by $1+1$.

But if each of these quotients be divided by $2-1$, the sum of each of the new quotients, when the terms are taken separately, will be greater than that of each of the former quotients. Thus,

$2-1)\ \ 2-\frac{3}{2}+\frac{7}{4}-\frac{13}{8}+\frac{27}{16}-\frac{53}{32}+\frac{107}{64}-\frac{213}{128}$, &c. $(1-\frac{1}{4}+\frac{6}{8}-\frac{7}{16}+\frac{20}{32}-\frac{33}{64}$, &c. $=\frac{3}{4}+\frac{5}{16}+\frac{7}{64}+\frac{9}{256}$, &c. $=1\frac{2}{9}$, instead of $\frac{2}{3}$.

$2-1)\ \ 3-\frac{5}{2}+\frac{11}{4}-\frac{21}{8}+\frac{43}{16}-\frac{85}{32}$, &c. $(\frac{3}{2}-\frac{1}{2}+\frac{9}{8}-\frac{12}{16}+\frac{31}{32}-\frac{54}{64}+\frac{117}{128}-\frac{224}{256}$, &c. $=1+\frac{6}{16}+\frac{8}{64}+\frac{10}{256}$, &c. $=1\frac{5}{9}$, instead of $\frac{2}{3}$.

And so on in other instances, in which it will be found that the sum of each succeeding will be greater than that of each preceding quotient, though the sums of the dividends are equal to each other, and the divisors are every where the same.

$$1+1)\ 1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\tfrac{1}{25}+\tfrac{1}{36},\ \&c.\ \left(1-\tfrac{3}{4}+\tfrac{31}{36}-\tfrac{469}{576}+\tfrac{3019}{3600}-\tfrac{2919}{3600}+\tfrac{48877}{58800}-\tfrac{191208}{235200},\ \&c.=\tfrac{1}{4}+\tfrac{1}{16}+\tfrac{1}{36},\ \&c.\right.$$

$$\begin{array}{l}
1+1 \\ \hline
-\tfrac{3}{4}+\tfrac{1}{9} \\
-\tfrac{3}{4}-\tfrac{3}{4} \\ \hline
\quad+\tfrac{31}{36}+\tfrac{1}{16} \\
\quad+\tfrac{31}{36}+\tfrac{31}{36} \\ \hline
\qquad-\tfrac{469}{576}+\tfrac{1}{25} \\
\qquad-\tfrac{469}{576}-\tfrac{469}{576} \\ \hline
\qquad\quad+\tfrac{3019}{3600}+\tfrac{1}{36} \\
\qquad\quad+\tfrac{3019}{3600}+\tfrac{3019}{3600} \\ \hline
\qquad\qquad-\tfrac{2919}{3600}+\tfrac{1}{49} \\
\qquad\qquad-\tfrac{2919}{3600}-\tfrac{2919}{3600} \\ \hline
\qquad\qquad\quad+\tfrac{48877}{58800}+\tfrac{1}{64} \\
\qquad\qquad\quad+\tfrac{48877}{58800}+\tfrac{48877}{58800} \\ \hline
\qquad\qquad\qquad-\tfrac{191208}{235200},\ \&c.
\end{array}$$

It is remarkable, in this instance, that the quotient, according to the *distributed* value, is $\frac{1}{4}$ of the dividend: for $\frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\frac{1}{64}$, &c. is $\frac{1}{4}$ of $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}$, &c., as is evident by dividing each of the terms by 4.

3—2) $1+1+1+1+1+1$, &c.

$1-\frac{2}{3}$

$+\frac{5}{3}+1$

$+\frac{5}{3}-\frac{10}{9}$

$+\frac{19}{9}+1$

$+\frac{19}{9}-\frac{38}{27}$

$+\frac{65}{27}$, &c.

($\frac{1}{3}+\frac{5}{9}+\frac{19}{27}+\frac{65}{81}+\frac{211}{243}$, &c. instead of $1+1+1+1+1$, &c.; from which series it differs by the series $\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}$, &c. $=\frac{2}{3-2}=2$.

4—3) $1+1+1+1+1+1$, &c. ($\frac{1}{4}+\frac{7}{16}+\frac{37}{64}+\frac{175}{256}+\frac{781}{1024}$, &c. instead of $1+1+1+1+1$, &c.; from which series it differs by the series $\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+\frac{81}{256}$, &c. $=\frac{3}{4-3}=3$.

5—4) $1+1+1+1+1+1$, &c. $\frac{1}{5}+\frac{9}{25}+\frac{61}{125}+\frac{369}{625}+\frac{2101}{3125}$, &c., instead of $1+1+1+1+1$, &c.; from which series it differs by the series $\frac{4}{5}+\frac{16}{25}+\frac{64}{125}+\frac{256}{625}$, &c. $=\frac{4}{5-4}=4$.

6—5) $1+1+1+1+1+1$, &c. ($\frac{1}{6}+\frac{11}{36}+\frac{91}{216}+\frac{671}{1296}+\frac{4651}{7776}$, &c. instead of $1+1+1+1+1$, &c., from which series it differs by the series $\frac{5}{6}+\frac{25}{36}+\frac{125}{216}+\frac{625}{1296}$, &c. $=\frac{5}{6-5}=5$.

2—1) $1+1+1+1+1$, &c. ($\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\frac{31}{32}$, &c. instead of $1+1+1+1+1+1$, &c.; from which series it is deficient by the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$, &c. $=1$.

3—1) $1+1+1+1+1+1+1$, &c.

$1-\frac{1}{3}$

$+\frac{4}{3}+1$

$\frac{4}{3}-\frac{4}{9}$

$+\frac{13}{9}+1$

$+\frac{13}{9}-\frac{13}{27}$

$+\frac{40}{27}+1$, &c.

($\frac{1}{3}+\frac{4}{9}+\frac{13}{27}+\frac{40}{81}$, &c. instead of $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$, &c., from which series it is deficient by the series $\frac{1}{6}+\frac{1}{18}+\frac{1}{54}+\frac{1}{162}$, &c. $=\frac{1}{4}$.

4—1) $1+1+1+1+1$, &c.

$1-\frac{1}{4}$

$+\frac{5}{4}+1$

$+\frac{5}{4}-\frac{5}{16}$

$+\frac{21}{16}+1$

$+\frac{21}{16}-\frac{21}{64}$

$+\frac{85}{64}+1$, &c.

($\frac{1}{4}+\frac{5}{16}+\frac{21}{64}+\frac{85}{256}$, &c. instead of $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}$, &c., from which series it is deficient by the series $\frac{1}{12}+\frac{1}{48}+\frac{1}{192}+\frac{1}{768}$, &c. $=\frac{1}{9}$.

$$\begin{array}{l}
5-1)\ 1+1+1+1+1+1, \&c.\ (\frac{1}{5}+\frac{6}{25}+\frac{31}{125}+\frac{156}{625}, \&c. \\
\quad 1-\frac{1}{5} \\
\hline
\quad +\frac{6}{5}+1 \\
\quad +\frac{6}{5}-\frac{6}{25} \\
\hline
\quad\quad +\frac{31}{25}+1 \\
\quad\quad +\frac{31}{25}-\frac{31}{125} \\
\hline
\quad\quad\quad +\frac{156}{125}+1, \&c.
\end{array}$$

instead of $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$, &c., from which series it is deficient by the series $\frac{1}{20}+\frac{1}{100}+\frac{1}{500}+\frac{1}{2500}=\frac{1}{16}$.

The difference between the two series, when the divisor is $6-1$, will be $\frac{1}{25}$; when it is $7-1$, $\frac{1}{36}$; and so on *ad infinitum*.

Again,

$$\begin{array}{l}
\frac{1}{2}-\frac{1}{4}+\frac{3}{8}-\frac{5}{16}+\frac{11}{32}-\frac{21}{64}, \&c.)\ 1\quad(2+1-1 \\
\quad 1-\frac{1}{2}+\frac{6}{8}-\frac{10}{16}+\frac{22}{32}-\frac{42}{64}, \&c. \\
\hline
\quad\cdot +\frac{1}{2}-\frac{6}{8}+\frac{10}{16}-\frac{22}{32}+\frac{42}{64}, \&c. \\
\quad +\frac{1}{2}-\frac{1}{4}+\frac{3}{8}-\frac{5}{16}+\frac{11}{32}, \&c. \\
\hline
\quad\quad\cdot -\frac{1}{2}+\frac{4}{16}-\frac{12}{32}+\frac{20}{64}, \&c. \\
\quad\quad -\frac{1}{2}+\frac{1}{4}-\frac{3}{8}+\frac{5}{16}, \&c. \\
\hline
\quad\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{array}$$

But if the divisor, by taking the terms separately, is the same as $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}$, &c. $=\frac{1}{3}$, the quotient ought to be 3, instead of $2+1-1$.

If the series $\frac{1}{2}-\frac{1}{4}+\frac{3}{8}-\frac{5}{16}+\frac{11}{32}-\frac{21}{64}$, &c., which, when the terms are taken separately, is equal to $\frac{1}{3}$, be multiplied by $\frac{1}{2}$, it will become $\frac{1}{4}-\frac{1}{8}+\frac{3}{16}-\frac{5}{32}+\frac{11}{64}-\frac{21}{128}$, &c. Let then 1 be divided by this, as below:

$$\begin{array}{l}
\frac{1}{4}-\frac{1}{8}+\frac{3}{16}-\frac{5}{32}+\frac{11}{64}-\frac{21}{128}, \&c.)\ 1\quad(4+2-2 \\
\quad 1-\frac{1}{2}+\frac{3}{4}-\frac{5}{8}+\frac{11}{16}-\frac{21}{32}, \&c. \\
\hline
\quad\cdot +\frac{1}{2}-\frac{3}{4}+\frac{5}{8}-\frac{11}{16}+\frac{21}{32}, \&c. \\
\quad +\frac{1}{2}-\frac{1}{4}+\frac{6}{16}-\frac{10}{32}+\frac{22}{64}, \&c. \\
\hline
\quad\quad\cdot -\frac{1}{2}+\frac{1}{4}-\frac{12}{32}+\frac{20}{64}, \&c. \\
\quad\quad -\frac{1}{2}+\frac{1}{4}-\frac{12}{32}+\frac{20}{64}, \&c. \\
\hline
\quad\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{array}$$

And the quotient will be 4, instead of 6.

But if the series $\frac{1}{2}-\frac{1}{4}+\frac{3}{8}-\frac{5}{16}$, &c. is multiplied by $\frac{1}{3}$, it will become $\frac{1}{6}-\frac{1}{12}+\frac{3}{24}-\frac{5}{48}+\frac{11}{96}-\frac{21}{192}$, &c.; and if it divides 1, the quotient will be as below:

$$\frac{1}{6}-\frac{1}{12}+\frac{3}{24}-\frac{5}{48}+\frac{11}{96}-\frac{21}{192},\ \&c.)\ 1\ \ (6+3-3$$
$$\begin{array}{r}
\underline{1-\frac{1}{2}+\frac{3}{4}-\frac{5}{8}+\frac{11}{16},\ \&c.}\\
+\frac{1}{2}-\frac{3}{4}+\frac{5}{8}-\frac{11}{16},\ \&c.\\
\underline{+\frac{1}{2}-\frac{1}{4}+\frac{9}{24}-\frac{15}{48},\ \&c.}\\
-\frac{1}{2}+\frac{1}{4}-\frac{9}{24},\ \&c.\\
\underline{-\frac{1}{2}+\frac{1}{4}-\frac{9}{24},\ \&c.}\\
\cdot\quad\cdot\quad\cdot
\end{array}$$

And will be 6, instead of 9.

$$\frac{1}{2}-\frac{1}{4}+\frac{3}{8}-\frac{5}{16}+\frac{11}{32}-\frac{21}{64},\ \&c.)\ \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64},\ \&c.\ (1+1$$
$$\begin{array}{r}
\underline{\frac{1}{2}-\frac{1}{4}+\frac{3}{8}-\frac{5}{16}+\frac{11}{32}-\frac{21}{64},\ \&c.}\\
+\frac{1}{2}-\frac{1}{4}+\frac{6}{16}-\frac{10}{32}+\frac{22}{64},\ \&c.\\
\underline{\frac{1}{2}-\frac{1}{4}+\frac{3}{8}-\frac{5}{16}+\frac{11}{32},\ \&c.}\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{array}$$

Here the quotient is $1+1$, instead of 3, and instead of $2+1-1$, which is the quotient when the dividend is 1, instead of $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, &c.

$$\frac{1}{4}-\frac{1}{8}+\frac{3}{16}-\frac{5}{32}+\frac{11}{64}-\frac{21}{128},\ \&c.)\ \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64},\ \&c.\ (2+2$$
$$\begin{array}{r}
\underline{\frac{1}{2}-\frac{1}{4}+\frac{6}{16}-\frac{10}{32}+\frac{22}{64}-\frac{42}{128},\ \&c.}\\
+\frac{1}{2}-\frac{1}{4}+\frac{3}{8}-\frac{5}{16}+\frac{11}{32},\ \&c.\\
\underline{+\frac{1}{2}-\frac{1}{4}+\frac{6}{16}-\frac{10}{32}+\frac{22}{64},\ \&c.}\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{array}$$

Here the quotient is $2+2$, instead of 6, and instead of $4+2-2$, which is the quotient when the dividend is 1, instead of $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$, &c.

It will likewise be found, that whatever whole number is prefixed to the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$, &c., the series, with the whole number so prefixed, will always produce, when it is divided by $1+1$, a quotient, the aggregate of the

terms of which, when they are taken separately, will be equal to $\frac{2}{3}$.

15. In the infinite series $1+2+3+4+5+6$, &c., the *distributed* value of all the odd exceeds the *distributed* value of all the even terms by the fraction $\frac{1}{1+2+1}$.

For the infinite series of odd terms $1+0+3+0+5+0+7$, &c., is produced by the expansion of $\frac{1+0+1}{1+0-2+0+1}$, and the infinite series of even terms $0+2+0+4+0+6+0+8$, &c. is produced by the expansion of $\frac{0+2}{1+0-2+0+1}$. But the difference between the latter and the former of these expressions is $\frac{1-2+1}{1+0-2+0+1}$; and this, by dividing both the numerator and denominator by $1-2+1$, will be equal to $\frac{1}{1+2+1}$.

Again, in the infinite series of triangular numbers $1+3+6+10+15+21+28$, &c., produced by the expansion of $\frac{1}{1-3+3-1}$, the difference between the *distributed* value of the terms $1+0+6+0+15+0+28+0+45$, &c., and the *distributed* value of the terms $0+3+0+10+0+21+0+36$, &c. is the fraction $\frac{1}{1+3+3+1}$. For the terms of the former series are produced by the expansion

of $\frac{1+0+3}{1+0-3+0+3+0-1}$, and the terms of the latter by the expansion of $\frac{0+3+0+1}{1+0-3+0+3+0-1}$; and the difference between the two is $\frac{1-3+3-1}{1+0-3+0+3+0-1}$, which, by dividing both the numerator and denominator by $1-3+3-1$, is equal to $\frac{1}{1+3+3+1}$.

But in the infinite series $1+4+10+20+35+56$, &c., produced by the expansion of $\frac{1}{1-4+6-4+1}$, the difference between the aggregate of the terms $1+10+35+84$, &c., and the aggregate of the terms $0+4+20+56$, &c. is $\frac{1}{1+4+6+4+1}$. And thus it appears that the differences between the aggregates of the alternate terms in the series produced by the expansion of the square, cube, biquadrat, &c. of the expression $\frac{1}{1-1}$, proceed in a geometrical ratio, being $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, &c. It is also remarkable that the denominators of these fractional differences consist of the same numbers as the expressions from which the above infinite series are produced. For the expressions are $\frac{1}{1-2+1}$, $\frac{1}{1-3+3-1}$, $\frac{1}{1-4+6-4+1}$, and the fractional differences are $\frac{1}{1+2+1}$, $\frac{1}{1+3+3+1}$, $\frac{1}{1+4+6+4+1}$.

16. Though in finite numbers one square cannot be found double of another, yet this can be effected in expressions which, when expanded, produce infinite series of whole numbers. Thus, for instance, as the expression $\frac{1}{1-2+1}$ is a square; for it is equal to $\frac{1}{1-1} \times \frac{1}{1-1}$; so likewise $\frac{1+1}{1-2+1}$, the double of it, is an infinite square. For this expression, when expanded, gives the series of odd numbers $1+3+5+7+9+11$, &c.; the sum of any number of which terms is a square, and therefore the sum of all the terms is a square. Hence the infinite series $\frac{1}{1-2+1} + \frac{1+1}{1-2+1} + \frac{1+2+1}{1-2+1} + \frac{4+4}{1-2+1} + \frac{16}{1-2+1} + \frac{16+16}{1-2+1} + \frac{64}{1-2+1}$, &c., will be an infinite series of squares double of each other. For a square multiplied by a square produces a square; and therefore $\frac{1+2+1}{1-2+1}$ will be a square double of the square $\frac{1+1}{1-2+1}$; because the numerator $1+2+1$ is a square, and is the double of the numerator $1+1$. Again, $\frac{4}{1-2+1}$ is a square, because it is equal to $\frac{1}{1-2+1} \times 4$; and therefore $\frac{1+1}{1-2+1}$, which is a square multiplied by 4, will

be equal to $\frac{4+4}{1-2+1}$, and which, consequently, will also be a square. In like manner, $\frac{16}{1-2+1}$ is a square, and therefore $\frac{1+1}{1-2+1} \times \frac{16}{1} = \frac{16+16}{1-2+1}$ will likewise be a square; and so of the rest.

In this series it is remarkable, that of every two squares, taken in their natural order, the root of the first only can be exactly obtained. Thus, of the squares $\frac{1}{1-2+1}$, $\frac{1+1}{1-2+1}$, the root of the former, but not of the latter, can be obtained. This is also the case with the squares $\frac{1+2+1}{1-2+1}$, $\frac{4+4}{1-2+1}$; and so of all the rest. But though the root of the first only of two such squares can be obtained, yet, on the contrary, the root of the sum of any number of the terms of which the second of these squares consists may be found; but this is not the case with the sum of any number of the terms of the first of these squares. Thus the terms of the square $\frac{1+1}{1-2+1}$ are $1+3+5+7+9$, &c.; and it is evident that the sum of any number of these terms is a square whose root can be found. But the terms of the square $\frac{1}{1-2+1}$ are $1+2+3+4+5+6$, &c., and it is evident that this will not be the case with these terms,

because not every number of them is a square. Thus, too, the terms of the square $\frac{4+4}{1-2+1}$ are $4+12+20+28$, &c., the root of the sum of any of which terms may be obtained; but this is not the case with the terms of the square $\frac{1+2+1}{1-2+1}$: for these are $1+4+8+12$, &c. And so of the rest.

17. In the First Book (parag. 21) it has been observed, that the series $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\frac{1}{7.8}+\frac{1}{9.10}$, &c., viz. $\frac{1}{2}+\frac{1}{12}+\frac{1}{30}+\frac{1}{56}+\frac{1}{90}+\frac{1}{132}+\frac{1}{182}+\frac{1}{240}$, &c., is the series discovered by Lord Brouncker for the quadrature of the hyperbola. If, therefore, each term of this series after the first term is doubled, the series will become $\frac{1}{2}+\frac{1}{6}+\frac{1}{15}+\frac{1}{28}+\frac{1}{45}+\frac{1}{66}+\frac{1}{91}+\frac{1}{120}$, &c. But this series, the first term excepted, consists of half the terms of the series of triangular reciprocals $\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\frac{1}{28}+\frac{1}{36}$, &c., as is evident from inspection. And the expression, by the expansion of which the series $1+\frac{1}{6}+\frac{1}{15}+\frac{1}{28}+\frac{1}{45}$, &c., minus the series $\frac{1}{3}+\frac{1}{10}+\frac{1}{21}+\frac{1}{36}$, &c. is produced, is, according to our method of notation, $\frac{1}{1+.1}=\frac{1}{1+\frac{1}{2}}$. For the series $1+\frac{1}{6}+\frac{1}{15}+\frac{1}{28}+\frac{1}{45}$, &c., added to the series $\frac{1}{3}+\frac{1}{10}+\frac{1}{21}+\frac{1}{36}$, &c.,

is equal to the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}$, &c. $=2$. But $\frac{1}{1+\frac{1}{2}}=\frac{2}{3}$; and therefore the difference between $1+\frac{1}{6}+\frac{1}{15}+\frac{1}{28}$, &c. and $\frac{1}{3}+\frac{1}{10}+\frac{1}{21}+\frac{1}{36}$, &c., is incommensurably $\frac{2}{3}$. The sum, however, of these two series is equal to 2, *i. e.* to $\frac{6}{3}$. Hence the former of these series is incommensurably $\frac{4}{3}$, and the latter is $\frac{2}{3}$. Hence, too, $\frac{1}{6}+\frac{1}{15}+\frac{1}{28}$, &c. is incommensurably $\frac{1}{3}$. But this series is the double of $\frac{1}{12}+\frac{1}{30}+\frac{1}{56}+\frac{1}{90}$, &c., and consequently this latter series is equal to $\frac{1}{6}$. The series, therefore, $\frac{1}{2}+\frac{1}{12}+\frac{1}{30}+\frac{1}{56}$, &c. is incommensurably equal to $\frac{1}{2}+\frac{1}{6}=\frac{8}{12}=\frac{2}{3}$.

18. It appears, from induction, to be universally true, that in all infinite series, produced by the expansion of expressions that have severally unity for their numerator, and unity connected by the affirmative sign with a simple fraction for their denominator, the sum of the terms connected with the affirmative sign will be to the sum of the terms connected with the negative sign, as the denominator is to the numerator of that simple fraction. Thus in the series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}$, &c., produced by the expansion of $\frac{1}{1+\frac{1}{2}}$, the aggregate of $1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}$, &c. is to the aggregate of $\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}$, &c. as 2 to 1.

Thus, too, in the series $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}-\frac{1}{243}$, &c. $= \frac{1}{1+\frac{1}{3}}$, the sum of $1+\frac{1}{9}+\frac{1}{81}$, &c. is to the sum of $\frac{1}{3}+\frac{1}{27}+\frac{1}{243}$, &c. as 3 to 1. And so in all other similar instances.

This likewise appears to be the case with fractional infinite series, whose aggregates are the same as those produced by the expansion of such expressions as the above, though the series themselves are not produced by such expressions. Thus, from what we have above shown, the ratio of the terms with the affirmative sign is to the ratio of the terms with the negative sign in the series $1-\frac{1}{3}+\frac{1}{6}-\frac{1}{10}+\frac{1}{15}$, &c., as 4 to 2, *i. e.* as 2 to 1; for the sum of the former is $\frac{4}{3}$, and of the latter $\frac{2}{3}$. And these are also the sums of the terms with the affirmative and negative signs in the series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}$, &c. Thus, too, the sums of the two series $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}$, &c. and $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}-\frac{1}{243}$, &c., are the same; for each is equal to $\frac{6}{8}$ or $\frac{3}{4}$. And in each the sum of the terms with the affirmative sign is $\frac{9}{8}$, and of the terms with the negative sign is $\frac{3}{8}$; and therefore the ratio is that of 3 to 1. And thus also the sums of the series $1-\frac{1}{7}+\frac{1}{49}-\frac{1}{343}+\frac{1}{2401}$, &c., produced by the expansion of

$\frac{1}{1+\frac{1}{7}}$, and of the series $1-\frac{1}{8}+\frac{1}{27}-\frac{1}{64}+\frac{1}{125}$, &c., are the same, each being equal to $\frac{7}{8}$. And in each the sum of the terms with the affirmative sign is $\frac{49}{8}$, and of the terms with the negative sign is $\frac{7}{8}$; and consequently the ratio is that of 7 to 1.

19. As $\frac{1}{1-3+3-1}$ is, when expanded, the infinite series of triangles, $1+3+6+10+15+21$, &c.; and as $\frac{1+1}{1-3+3-1}$ gives the infinite series of squares, $1+4+9+16+25+36$, &c.: so $\frac{1+2}{1-3+3-1}$ will give the infinite series of pentagons, $1+5+12+22+35$, &c.; $\frac{1+3}{1-3+3-1}$ the infinite series of hexagons, $1+6+15+28+45+66$, &c.; $\frac{1+4}{1-3+3-1}$ the series of heptagons, $1+7+18+34$, &c.; $\frac{1+5}{1-3+3-1}$ the series of octagons, $1+8+21+40$, &c.; $\frac{1+6}{1-3+3-1}$ the series of enneagons, $1+9+24+46$, &c.; and $\frac{1+7}{1-3+3-1}$ the infinite series of decagons, $1+10+27+52+85$, &c. Hence it follows that infinite series of polygonous numbers will be to each other in the ratio of the natural series of numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c.

20. The fractional reciprocals also of these poly-

gonous numbers will be to each other as 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$, &c., as is evident from our method of notation. For $\frac{1}{1-.1}$, when expanded, is $1+.1+{}^{2}1+{}^{3}1+{}^{4}1+{}^{5}1+{}^{6}1$, &c. $=1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}$, &c., the triangular reciprocals; $\frac{1}{1-{}^{2}1}$ is $1+{}^{2}1+{}^{4}1+{}^{6}1+{}^{8}1$, &c. $=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}$, &c., the reciprocals of the squares; $\frac{1}{1-{}^{3}1}$ is $1+{}^{3}1+{}^{6}1+{}^{9}1+{}^{12}1$, &c. $=1+\frac{1}{5}+\frac{1}{12}+\frac{1}{22}+\frac{1}{35}$, &c. the pentagonal reciprocals; $\frac{1}{1-{}^{4}1}$ is $1+{}^{4}1+{}^{8}1+{}^{12}1+{}^{16}1$, &c. $=1+\frac{1}{6}+\frac{1}{15}+\frac{1}{28}+\frac{1}{45}+\frac{1}{66}$, &c., the hexagonal reciprocals; $\frac{1}{1-{}^{5}1}$ is $1+{}^{5}1+{}^{10}1+{}^{15}1+{}^{20}1$, &c. $=1+\frac{1}{7}+\frac{1}{18}+\frac{1}{34}$, &c., the heptagonal reciprocals; $\frac{1}{1-{}^{6}1}$ is $1+{}^{6}1+{}^{12}1+{}^{18}1+{}^{24}1+{}^{30}1$, &c. $=1+\frac{1}{8}+\frac{1}{21}+\frac{1}{40}$, &c. the reciprocals of the octagons; $\frac{1}{1-{}^{7}1}$ is $1+{}^{7}1+{}^{14}1+{}^{21}1+{}^{28}1$, &c. $=1+\frac{1}{9}+\frac{1}{24}+\frac{1}{46}$, &c. the enneagonal reciprocals; and $\frac{1}{1-{}^{8}1}$ is $1+{}^{8}1+{}^{16}1+{}^{24}1+{}^{32}1$, &c. $=1+\frac{1}{10}+\frac{1}{27}+\frac{1}{52}+\frac{1}{85}$, &c., the decagonal reciprocals. But $\frac{1}{1-.1}=1+1$; $\frac{1}{1-{}^{2}1}=1\frac{1}{2}$; $\frac{1}{1-{}^{3}1}=1\frac{1}{3}$; $\frac{1}{1-{}^{4}1}=1\frac{1}{4}$; $\frac{1}{1-{}^{5}1}=1\frac{1}{5}$; $\frac{1}{1-{}^{6}1}=1\frac{1}{6}$; $\frac{1}{1-{}^{7}1}=1\frac{1}{7}$; and $\frac{1}{1-{}^{8}1}=1\frac{1}{8}$. All these fractional re-

ciprocals, however, are incommensurable quantities in their aggregates: for the expressions, by the expansion of which they are produced, are less than the aggregate values of the series.

21. In the following instances the numerators of the expressions which, when expanded, produce infinite series, have a variable value:

$\frac{1+1}{1-2} = 1+3+6+12+24$, &c. $\frac{\frac{3}{2}}{1-2} = \frac{3}{2}+3+6+12+24$, &c.

$\frac{1+1}{1-3} = 1+4+12+36+108$, &c. $\frac{\frac{4}{3}}{1-3} = \frac{4}{3}+4+12+36+108$, &c.

$\frac{1+1}{1-4} = 1+5+20+80+320$, &c. $\frac{\frac{5}{4}}{1-4} = \frac{5}{4}+5+20+80+320$, &c.

And so in other instances *ad infinitum*, such as $\frac{1+1}{1-5}$, $\frac{1+1}{1-6}$, &c. Hence it appears that $1+1$, in these instances, has a variable value; and that when it is the numerator of the expression producing the series $1+3+6+12+24$, &c., it is equal to $\frac{3}{2}$, an infinitesimal excepted: for $\frac{1}{2}$ is but an infinitesimal of this series. In the expression $\frac{1+1}{1-3}$, $1+1$ is equal to $\frac{4}{3}$, an infinitely small part of the series $1+4+12+36+108$, &c. excepted:

for such is $\frac{2}{3}$. In the expression $\frac{1+1}{1-4}$, $1+1$ is equal to $\frac{5}{4}$, an infinitesimal excepted: and so on.

The expressions, also, which produce the fractional reciprocals of these series, will be $\frac{\frac{2}{3}}{2-1} = \frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\frac{1}{24}$, &c., $\frac{\frac{3}{4}}{3-1} = \frac{1}{4}+\frac{1}{12}+\frac{1}{36}+\frac{1}{108}$, &c., and $\frac{\frac{4}{5}}{4-1} = \frac{1}{5}+\frac{1}{20}+\frac{1}{80}+\frac{1}{320}$, &c. And as $\frac{\frac{3}{2}}{1-2}$ is to $\frac{1}{1-2}$, so is $\frac{1}{2-1}$ to $\frac{\frac{2}{3}}{2-1}$. Likewise, as $\frac{\frac{4}{3}}{1-3} : \frac{1}{1-3} :: \frac{1}{3-1} : \frac{\frac{3}{4}}{3-1}$. And as $\frac{\frac{5}{4}}{1-4} : \frac{1}{1-4} :: \frac{1}{4-1} : \frac{\frac{4}{5}}{4-1}$. And so in all other similar instances.

Again,

$$\frac{1+2}{1-2} = 1+4+8+16+32, \text{ \&c.}$$

$$\frac{1+2}{1-3} = 1+5+15+45+135, \text{ \&c.}$$

$$\frac{1+2}{1-4} = 1+6+24+96+384, \text{ \&c.}$$

$$\frac{\frac{4}{2}}{1-2} = \frac{4}{2}+4+8+16+32, \text{ \&c.}$$

$$\frac{\frac{5}{3}}{1-3} = \frac{5}{3}+5+15+45+135, \text{ \&c.}$$

$$\frac{\frac{6}{4}}{1-4} = \frac{6}{4}+6+24+96+384, \text{ \&c.}$$

Here $1+2$ is variable, in the same manner as $1+1$ in the former instances.

The expressions, likewise, which produce the fractional reciprocals of these series, will be,

$$\frac{\frac{1}{2}}{2-1} = \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32}, \text{ \&c.}$$

$$\frac{\frac{1}{3}}{3-1} = \tfrac{1}{5} + \tfrac{1}{15} + \tfrac{1}{45} + \tfrac{1}{135}, \text{ \&c.}$$

$$\frac{\frac{1}{6}}{4-1} = \tfrac{1}{6} + \tfrac{1}{24} + \tfrac{1}{96} + \tfrac{1}{384}, \text{ \&c.}$$

And the analogy between these reciprocals, and the series of whole numbers of which they are the reciprocals, will be the same as in the former instances.

It is remarkable, in these expressions, when the numerators are $1+1$, $1+2$, $1+3$, &c., that the fractional numerators, to which the integral numerators are equivalent, are produced invariably as follows: Add the second term of the integral numerator to the second term of the denominator, and the sum will be the numerator of the fractional numerator, and the said second term of the denominator will be the denominator of the fractional numerator.

Thus, in the expression $\frac{1+1}{1-2}$, which is equivalent (an infinitesimal excepted) to $\frac{\frac{3}{2}}{1-2}$, $1+2$ is equal to 3, the numerator of $\frac{3}{2}$; and the 2 of $1-2$

is the denominator of $\frac{3}{2}$. Thus, too, in $\frac{1+1}{1-3}$, which is equivalent to $\frac{\frac{4}{3}}{1-3}$, the numerator 4 is equal to $1+3$, and the denominator of it is the 3 of $1-3$. And so of the rest.

In the following expressions also the numerators will be found to be variable, viz. $\frac{2+1}{1-2}$, $\frac{3+1}{1-2}$, $\frac{4+1}{1-2}$, $\frac{5+1}{1-2}$, &c., and to be equivalent to $\frac{\frac{5}{2}}{1-2}$, $\frac{\frac{7}{2}}{1-2}$, $\frac{\frac{9}{2}}{1-2}$, $\frac{\frac{11}{2}}{1-2}$, &c. And the analogy, with their fractional reciprocals, will be the same as in the preceding instances.

The like will also take place in the following instances:

$$\frac{2-1}{1-2} = 2+3+6+12+24+48, \text{ \&c.}$$

$$\frac{3-1}{1-2} = 3+5+10+20+40, \text{ \&c.}$$

$$\frac{2-1}{1-3} = 2+5+15+45+135, \text{ \&c.}$$

$$\frac{3-1}{1-3} = 3+8+24+72+216, \text{ \&c.}$$

For these will be equivalent, an infinitesimal excepted, to

$$\frac{\frac{3}{2}}{1-2} = \tfrac{3}{2}+3+6+12+24, \text{ \&c.}$$

$$\frac{\frac{5}{2}}{1-2} = \tfrac{5}{2}+5+10+20+40, \text{ \&c.}$$

$$\frac{\frac{5}{3}}{1-3} = \tfrac{5}{3} + 5 + 15 + 45 + 135, \text{ \&c.}$$

$$\frac{\frac{8}{3}}{1-3} = \tfrac{8}{3} + 8 + 24 + 72 + 216, \text{ \&c.}$$

Their fractional reciprocals also will be,

$$\frac{\frac{2}{3}}{2-1} = \tfrac{1}{3} + \tfrac{1}{6} + \tfrac{1}{12} + \tfrac{1}{24}, \text{ \&c.}$$

$$\frac{\frac{2}{5}}{2-1} = \tfrac{1}{5} + \tfrac{1}{10} + \tfrac{1}{20} + \tfrac{1}{40}, \text{ \&c.}$$

$$\frac{\frac{3}{5}}{3-1} = \tfrac{1}{5} + \tfrac{1}{15} + \tfrac{1}{45} + \tfrac{1}{135}, \text{ \&c.}$$

$$\frac{\frac{3}{8}}{3-1} = \tfrac{1}{8} + \tfrac{1}{24} + \tfrac{1}{72} + \tfrac{1}{216}, \text{ \&c.}$$

It is remarkable of the numerators in these instances, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{8}{3}$, &c., that 3, 5, and 8, &c., are the second terms of the series $2+3+6+12$, &c., $3+5+10+20$, &c., $3+8+24+72$, &c.

And in the last place, in the following instances the variableness of the numerators is manifest :

$$\frac{1-\frac{1}{2}}{1-2} = 1 + \tfrac{3}{2} + 3 + 6 + 12 + 24, \text{ \&c.}$$

$$\frac{\frac{3}{4}}{1-2} = \tfrac{3}{4} + \tfrac{3}{2} + 3 + 6 + 12, \text{ \&c.}$$

$$\frac{\frac{4}{3}}{2-1} = \tfrac{2}{3} + \tfrac{1}{3} + \tfrac{1}{6} + \tfrac{1}{12} + \tfrac{1}{24}, \text{ \&c.}$$

$$\frac{1-\frac{1}{3}}{1-2} = 1 + \tfrac{5}{3} + \tfrac{10}{3} + \tfrac{20}{3} + \tfrac{40}{3} + \tfrac{80}{3}, \text{ \&c.}$$

$$\frac{\frac{5}{6}}{1-2} = \tfrac{5}{6} + \tfrac{5}{3} + \tfrac{10}{3} + \tfrac{20}{3}, \text{ \&c.}$$

$$\frac{\frac{6}{5}}{2-1} = \tfrac{3}{5} + \tfrac{3}{10} + \tfrac{3}{20} + \tfrac{3}{40}, \text{ \&c.}$$

$$\frac{1-\frac{1}{4}}{1-2} = 1 + \tfrac{7}{4} + \tfrac{14}{4} + \tfrac{28}{4} (= 7) + 14 + 28, \text{ \&c.}$$

$$\frac{\frac{7}{8}}{1-2} = \tfrac{7}{8} + \tfrac{7}{4} + \tfrac{14}{4} + 7 + 14, \text{ \&c.}$$

$$\frac{\frac{8}{7}}{2-1} = \tfrac{4}{7} + \tfrac{4}{14} + \tfrac{1}{7} + \tfrac{1}{14} + \tfrac{1}{28}, \text{ \&c.}$$

$$\frac{1-\frac{1}{2}}{1-3} = 1 + \tfrac{5}{2} + \tfrac{15}{2} + \tfrac{45}{2} + \tfrac{135}{2}, \text{ \&c.}$$

$$\frac{\frac{5}{6}}{1-3} = \tfrac{5}{6} + \tfrac{5}{2} + \tfrac{15}{2} + \tfrac{45}{2}, \text{ \&c.}$$

$$\frac{\frac{6}{5}}{3-1} = \tfrac{3}{5} + \tfrac{3}{15} + \tfrac{3}{45} + \tfrac{3}{135}, \text{ \&c.}$$

APPENDIX.

APPENDIX.

The following particulars respecting infinite series, amicable and perfect numbers, &c., are, I believe, for the most part entirely new; and for this reason, and also because I am persuaded they will not be deemed by mathematicians to be uninteresting, they are inserted as an adjunct to this elementary work.

1. The series $\frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} + \frac{1}{3\,.\,4} + \frac{1}{4\,.\,5}$, &c. $= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20}$, &c. $= 1$. For, by multiplying each of the terms by 2, the denominator of the first term, we shall have $\frac{2}{2} + \frac{2}{6} + \frac{2}{12} + \frac{2}{20}$, &c. $= 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10}$, &c. $= 2$.

2. Again, $\frac{1}{1\,.\,2\,.\,3} + \frac{1}{2\,.\,3\,.\,4} + \frac{1}{3\,.\,4\,.\,5} + \frac{1}{4\,.\,5\,.\,6} + \frac{1}{5\,.\,6\,.\,7}$, &c. $= \frac{1}{6} + \frac{1}{24} + \frac{1}{60} + \frac{1}{120} + \frac{1}{210}$, &c. $= \frac{1}{4}$. For if each of the terms is multiplied by 6, the denominator of the first term, the series $\frac{6}{6} + \frac{6}{24} + \frac{6}{60} + \frac{6}{120} + \frac{6}{210}$, &c., will be produced, $= 1 + \frac{1}{4} + \frac{1}{10} + \frac{1}{20} + \frac{1}{35}$, &c. $= \frac{3}{2}$; and the $\frac{1}{6}$ of $\frac{3}{2} = \frac{3}{12} = \frac{1}{4}$.

3. Thus, too, $\frac{1}{1\,.\,2\,.\,3\,.\,4}+\frac{1}{2\,.\,3\,.\,4\,.\,5}+\frac{1}{3\,.\,4\,.\,5\,.\,6}+\frac{1}{4\,.\,5\,.\,6\,.\,7}+\frac{1}{5\,.\,6\,.\,7\,.\,8}$, &c. $=\frac{1}{24}+\frac{1}{120}+\frac{1}{360}+\frac{1}{840}+\frac{1}{1680}$, &c. $=\frac{1}{18}$. For, by multiplying each of the terms by 24, the denominator of the first term, we shall have the series $\frac{24}{24}+\frac{24}{120}+\frac{24}{360}+\frac{24}{840}+\frac{24}{1680}$, &c. $=1+\frac{1}{5}+\frac{1}{15}+\frac{1}{35}+\frac{1}{70}$, &c. $=\frac{4}{3}$; and the $\frac{1}{24}$ of $\frac{4}{3}=\frac{4}{172}=\frac{1}{18}$.

4. Again, $\frac{1}{1\,.\,2\,.\,3\,.\,4\,.\,5}+\frac{1}{2\,.\,3\,.\,4\,.\,5\,.\,6}+\frac{1}{3\,.\,4\,.\,5\,.\,6\,.\,7}+\frac{1}{4\,.\,5\,.\,6\,.\,7\,.\,8}$, &c.$=\frac{1}{120}+\frac{1}{720}+\frac{1}{2520}+\frac{1}{6720}$, &c. $=\frac{1}{96}$. For, by multiplying each of the terms by 120, the denominator of the first term, the series $1+\frac{1}{6}+\frac{1}{21}+\frac{1}{56}$, &c. will be produced $=\frac{5}{4}$; and the $\frac{1}{120}$ part of $\frac{5}{4}=\frac{5}{480}=\frac{1}{96}$.

5. And the like will take place *ad infinitum*, in all the series whose denominators are produced by the multiplication of the terms of the natural series, 1 . 2 . 3 . 4 . 5 . 6, &c., conformably to the preceding instances.

6. Farther still; if each term of the series $\frac{1}{2\,.\,4}+\frac{1}{4\,.\,6}+\frac{1}{6\,.\,8}+\frac{1}{8\,.\,10}$, &c. $=\frac{1}{8}+\frac{1}{24}+\frac{1}{48}+\frac{1}{80}$, &c. is multiplied by 8, the denominator

of the first term, the series will become $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$, &c. $=2$; and therefore $\frac{1}{8}+\frac{1}{24}+\frac{1}{48}$, &c. $=\frac{2}{8}=\frac{1}{4}$.

7. Again, if each term of the series $\frac{1}{3\,.\,6}+\frac{1}{6\,.\,9}+\frac{1}{9\,.\,12}+\frac{1}{12\,.\,15}$, &c. $=\frac{1}{18}+\frac{1}{54}+\frac{1}{108}+\frac{1}{180}$, &c. is multiplied by 18, the denominator of the first term, the series will become $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c. $=2$; and therefore the series $\frac{1}{18}+\frac{1}{54}+\frac{1}{108}+\frac{1}{180}$, &c., will be equal to $\frac{2}{18}=\frac{1}{9}$.

8. If also each term of the series $\frac{1}{4\,.\,8}+\frac{1}{8\,.\,12}+\frac{1}{12\,.\,16}+\frac{1}{16\,.\,20}$, &c. is multiplied by 32, the denominator of the first term, the series will become $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c.; and, consequently, $\frac{1}{32}+\frac{1}{96}+\frac{1}{192}$, &c. $=\frac{1}{4\,.\,8}+\frac{1}{8\,.\,12}+\frac{1}{12\,.\,16}$, &c. will be equal to $\frac{2}{32}=\frac{1}{16}$.

And in all similar series, each term being multiplied by the denominator of the first term of the series, will produce the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c. It also deserves to be remarked, that the aggregates of the several series will be the series $\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}$, &c.

9. Iamblichus, in his treatise On the Arith-

metic of Nicomachus, observes, (p. 47,) "that certain numbers were called amicable * by those who assimilated the virtues and elegant habits to numbers." He adds, "that 284 and 220 are numbers of this kind; for the parts of each are generative of each other, according to the nature of friendship, as was shown by Pythagoras. For some one asking him what a friend was, he answered, *another I*, (ετερος εγω); which is demonstrated to take place in these numbers." And he concludes with informing us, "that he shall discuss, in its proper place, what is delivered by the Pythagoreans relative to this most splendid and elegant theory." Unfortunately, he has not resumed this subject in the above-mentioned treatise, nor in any work of his that is extant; and the only writer I am acquainted with, who has written more fully concerning these numbers, is Ozanam, who, in his Mathematical Recreations, p. 15, observes of them as follows: "The two numbers 220 and 284 are called amicable;

* See my Theoretic Arithmetic; in which the reader will find many other interesting and novel particulars respecting numbers, besides what are contained in the following pages.

because the first, 220, is equal to the sum of the aliquot parts of the latter, 284, viz. $1+2+4+71+142=220$; and reciprocally, the latter, 284, is equal to the sum of the aliquot parts of the former, 220, viz. $1+2+4+5+10+11+20+22+44+55+110=284$.

"To find all the amicable numbers in order, make use of the number 2, which is of such a quality, that if you take 1 from its triple 6, from its sextuple 12, and from the octodecuple of its square, viz. from 72, the remainders are the three prime numbers 5, 11, and 71; of which, 5 and 11 being multiplied together, and the product 55 being multiplied by 4, the double of the number 2, this second product, 220, will be the first of the two numbers we look for. And to find the other 284, we need only to multiply the third prime number 71 by 4, the same double of 2 that we used before.

"To find two other amicable numbers, instead of 2 we make use of one of its powers that possesses the same quality, such as its cube 8. For, if you subtract an unit from its triple 24, from its sextuple 48, and from 1152, the octodecuple of its square 64, the remainders are

the three prime numbers, viz. 23, 47, 1151; of which the two first, 23, 47, ought to be multiplied together, and their product, 1081, ought to be multiplied by 16, the double of the cube 8, in order to have 17296 for the first of the two numbers demanded. And for the other amicable number, which is 18416, we must multiply the third prime number 1151 by 16, the same double of the cube 8.

"If you still want other amicable numbers, instead of 2, or its cube 8, make use of its square cube 64; for it has the same quality, and will answer as above."

Thus far Ozanam. But the two amicable numbers produced from 64, which he has omitted in this extract, are 9363584 and 9437056; and the three prime numbers, which are the remainders, are 191, 383, and 73727.

10. In the arithmetical series $1+2+3+4+5$, &c., when the number of terms is even and finite, the sum of the two middle terms, multiplied by the number of terms, is equal to twice the sum of the series. Thus in four terms, $3+2 = 5$ and $5\times 4 = 20$, which is the double of 10, the sum of the series.

11. In the geomet … 1 + 2 + 4 + 32, &c., when the … $\frac{1}{2}$ + 1 the sum of the seri … this to the double of th … es of 1 + 2 + 1 is the … + 4 + 1 is the doub…

12. In the geometrical series … 81, &c., the triple of the last term exceeds the double of the sum of the series by unity. Thus, in two terms, the triple of 3, *i. e.* 9, exceeds the double of the sum 1 + 3, *i. e.* 4 by 1. Thus the triple of 9, *i. e.* 27, exceeds the double of the sum 1 + 3 + 9, *i. e.* 13 by 1; and so of the rest.

13. In the geometrical series 1 + 4 + 16 + 64 + 256, &c., the quadruple of the last term exceeds the triple of the sum of the series by unity.

14. And in the geometrical series 1 + 5 + 25 + 125 + 625, &c., the quintuple of the last term exceeds the quadruple of the sum of the series by unity.

15. Again, in the series 1 + 2 + 4 + 8 + 16, &c., when the number of terms is finite, the last term, added to the last term less by unity, is equal to the sum of the series. Thus 2 added to 2,

the three = 1 + 2; 4 added to 4, less by 1 = 1 + which the, &c.

16. In the series 1 + 3 + 9 + 27 + 81, &c., if unity is subtracted from the last term, and the remainder, divided by 2, is added to the last term, the sum is equal to the sum of the series. Thus 3 − 1 = 2; this divided by 2 = 1, and 1 + 3 = 4 = the sum of the two first terms; 9 − 1 = 8, and 8 divided by 2 = 4, and 4 + 9 = 13 = 1 + 3 + 9, &c.

17. In the series 1 + 4 + 16 + 64 + 256, &c., if unity is subtracted from the last term, and the remainder, divided by 3, is added to the last term, the sum is equal to the sum of the series.

18. In the series 1 + 5 + 25 + 125 + 625, &c., if unity is subtraced from the last term, and the remainder, divided by 4, is added to the last term, the sum is equal to the sum of the series.

19. In the geometrical fractional series

$$\begin{array}{l} \frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}, \text{ \&c.} \\ \frac{1}{1}+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}, \text{ \&c.} \\ \frac{1}{1}+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}, \text{ \&c.} \\ \frac{1}{1}+\frac{1}{5}+\frac{1}{25}+\frac{1}{125}, \text{ \&c.,} \end{array}$$

the last term, multiplied by the sum of the denominators, is equal to the sum of the series, when

the number of terms is finite. Thus $\frac{1}{8} \times 1+2+4+8=\frac{15}{8}$, the sum of $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$. Thus $\frac{1}{27}+1+3+9+27=\frac{40}{27}$; and so of the rest. And this will be case whatever the ratio of the series may be.

And in the first of these series, if unity be taken from the denominator of the last term, and the remainder be added to the denominator, the sum arising from this addition, multiplied by the last term, will be equal to the sum of the series. Thus $8-1=7$ and $7+8=15$, and $\frac{1}{8} \times 15=\frac{15}{8}$, the sum of the series. In the second of these series, if unity be subtracted from the denominator of the last term, the remainder be divided by 2, and the quotient be added to the said denominator, the sum, multiplied by the last term, will be equal to the sum of the series. In the third series, after the subtraction, the remainder must be divided by 3; in the fourth by 4; in the fifth by 5; in the sixth by 6; and so on.

20. In the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c., which, infinitely continued, is equal to 2, an infinitesimal excepted, if any finite number of terms is assumed, then, if the denominator of the last term

be added to the denominator of the term immediately preceding it, and the sum of the two be multiplied by the last term, the product will be equal to the sum of the series. Thus in two terms, $1+3\times\frac{1}{3}=\frac{4}{3}=$ the sum of $1+\frac{1}{3}$. If there are three terms, then $3+6\times\frac{1}{6}=\frac{9}{6}=1+\frac{1}{3}+\frac{1}{6}$. If there are four terms, then $6+10\times\frac{1}{10}=\frac{16}{10}=1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$; and so of the rest.

If the half of each term of this series is taken, so as to produce the series $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}$, &c., then the half of the sum of the denominator of the last term, and the denominator of the term immediately preceding the last, multiplied by the last term, will be equal to the sum of the series. If the third of each term is taken, the sum of the two denominators must be divided by 3. If the fourth of each term is taken, the division must be by 4; and so of the rest.

21. If the half of each term of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, &c. is taken, so as to produce the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$, &c., and if the sum of the denominators is divided by 2, the quotient, multiplied by the last term, is equal to the sum of the series. Thus, $\frac{2+4+8}{2}\times\frac{1}{8}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$,

$\frac{2+4+8+16}{2} \times \frac{1}{16} = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$; and so of the rest.

But if the third of each term of this series is taken, the divisor must be 3; if the fourth is taken, the divisor must be 4; if the fifth, it must be 5; if the sixth, 6; and so on.

22. Imperfectly amicable numbers are such as 27 and 35, 39 and 55, 65 and 77, 51 and 91, 95 and 119, 69 and 133, 115 and 187, 87 and 247. For the parts of $27 = 3 \times 9$ are 1, 3, 9, the sum of which is 13; and the parts of $35 = 5 \times 7$ are 1, 5, 7; the sum of which is also 13.

The parts of $39 = 3 \times 13$ are 1, 3, 13; and the sum of these is 17. And this is also the sum of 1, 5, 11, the parts of $55 = 5 \times 11$.

The parts of $65 = 13 \times 5$ are 1, 5, 13. The parts of 77 are 1, 7, 11; and the sum of each is 19.

The parts of $51 = 17 \times 3$ are 1, 3, 17. The parts of $91 = 13 \times 7$ are 1, 7, 13; and the sum of each is 21. And so of the rest.

The sums of the numbers, from the multiplication of which these imperfectly amicable numbers are formed, are 12, 16, 18, 20, 24, 26, 28, 32; of which the first differs from the second

by 4, the second from the third, and the third from the fourth, by 2. And again, the fourth differs from the fifth by 4, the fifth from the sixth by 2, the sixth from the seventh by 2, and the seventh from the eighth by 4. Hence the differences are 2, 2, 4, 2, 2, 4; and so on *ad infin.*

Numbers of this kind are not noticed by any writers on arithmetic that I am acquainted with. I call them imperfectly amicable numbers; because, in two perfectly amicable numbers, the aggregate of the parts of the one is equal to the other; but in these the aggregate of the parts of one number is equal to the aggregate of the parts of the other; but the sum of each is less than the whole number. In perfectly amicable numbers, therefore, the parts of the one embosom, as it were, the whole of the other; but in these the parts of the one do not embosom the whole, but only a part of the other, because the aggregate of the parts falls short of the whole.

Hence, in numbers that are deficient, or the sum of whose parts is less than the numbers themselves, there is a great abundance of numbers; the sums of the parts of two of which are equal to each other. But in super-perfect

numbers, or such the sum of whose parts exceeds the numbers of which they are the parts, the numbers are very rare that have the above-mentioned property of deficient numbers. Thus, between 12 and 144 there are only two numbers, the sums of the parts of which are equal to each other, and these are 80 and 104. For the sum of the parts of each is 106.

As perfectly amicable numbers also adumbrate perfect friendship, and which consequently is founded in virtue, so these numbers are perspicuous images of the friendship subsisting among vicious characters; such of them whose parts are less than the whole, adumbrating the friendship between those who fall short of the medium in which true virtue consists; and those whose parts are greater than the whole, exhibiting an image of the friendship of such as exceed this medium. As likewise, of the vicious characters situated on each side of the medium, those that exceed it are more allied to virtue than those that fall short of it; and being more allied to virtue, are more excellent; and being more excellent, are more rarely to be found: thus, also,

in these numbers, the pairs whose parts are less than the whole numbers, are far more numerous than those whose parts are greater than the wholes of which they are the parts.

23. The series of unevenly-even numbers is as follows: 12, 20, 24, 28, 36, 40, 44, 48, 52, 56, 60, 68, 72, 76, 80, 84, 88, 92, 96, 100, 104, 108, 112, 116, 120, 124, 132, 136, 140, 144, 148, 152, 156, 160, 164, 168, 172, 176, 180, 184, 188, 192, 196, 200, 204, 208, 212, 216, 220, 224, 228, 232, 236, 240, 244, 248, 252, 260, &c.

In this series it is observable, in the first place, that the difference between the terms is every where either 8 or 4. Thus the difference between 12 and 20 is 8; but between 20 and 24, and 24 and 28, is 4. Again, the difference between 28 and 36 is 8; but between 36 and 40, 40 and 44, 44 and 48, 48 and 52, 52 and 56, 56 and 60, is 4. And again, the difference between 60 and 68 is 8; but between 68 and 72 is 4; and so on till we arrive at 124 and 132, the difference between which is 8.

In the second place, it is observable that the difference between 12 and 20 is 8; between 20

and 36 is 16; between 36 and 68 is 32; between 68 and 132 is 64, and so on; which differences are in a duple ratio.

In the third place, if the number which exceeds its preceding number by 8 is added to the number immediately preceding that which is so exceeded, the sum will be the following number immediately preceding that which exceeds by 8. Thus $36+24=60$, the number immediately preceding 68. Thus, too, $68+56=124$, which immediately precedes 132; and $132+120=252$; and so of the rest. From all which it is evident that the series of unevenly-even numbers, *ad infinitum,* may be easily obtained.

24. The sum of the parts of each term of the duple series 2, 4, 8, 16, 32, 64, &c., is equal to the whole of the term less by unity. Thus the part of 2 is 1; the parts of 4 are 2 and 1, the sum of which is 3; the parts of 8 are 4, 2, and 1, the aggregate of which is 7; and the parts of 16 are 8, 4, 2, 1, the sum of which is 15; and so of the rest.

25. But each of the terms of the triple series 3, 9, 27, 81, 243, &c., exceeds the double of the sum of its parts by unity. Thus the only part of

3 is 1; and 3 exceeds 2 by 1. The parts of 9 are 3 and 1, and 9 exceeds the double of the aggregate of these, *i. e.* 8, by 1. Thus, too, 27 exceeds the double of 13, the aggregate of its parts, 9, 3, and 1, by 1. Thus, also, 81 exceeds the double of 40, the sum of its parts, 27, 9, 3, and 1, by 1; and thus 243 exceeds twice 121, the aggregate of its parts 81, 27, 9, 3, and 1, by 1; and so of the rest.

26. In the quintuple series 5, 25, 125, 625, &c., it will be found that each term of the series exceeds the quadruple of the sum of its parts by unity.

27. In the septuple series each term exceeds the sextuple of the sum of its parts by unity.

28. In the noncuple series each term exceeds the octuple of the sum of its parts by unity; and thus, in all series formed by the multiplication of odd numbers, each term will exceed the sum arising from the multiplication of its parts by the odd number, by unity.

29. If to each term of the series,

	1,	2,	4,	8,	16,	32,	64,	128,	256,
6 is added, viz.	6,	6,	6,	6,	6,	6,	6,	6,	6,
the sums will be	7,	8,	10,	14,	22,	38,	70,	134,	262.

And if to the series	1, 3, 9, 27, 81, 243,	729, 2187, 6561,
be added	3, 6, 6, 6, 6, 6,	6, 6, 6,
the sums will be	4, 9, 15, 33, 87, 249,	735, 2193, 6567.

In which it is remarkable, that the aggregates of the parts of the five sums, 8, 10, 14, 22, 38, are 7, 8, 10, 14, 32; and of the parts of the five sums, 9, 15, 33, 87, 249, are 4, 9, 15, 33, 87. The aggregates also of the parts of the sums, 134, 262, are 70, 134; but this will not be the case with the sums beyond 262, if the terms of the duple series are continued, and 6 is added to them. Nor will it be the case with the sums of the triple series beyond 249.

30. In the following series of terms, arising from the multiplication of unevenly-even numbers, by the sums produced by the addition of them, perfect numbers also are contained.

This series consists of the terms 1, 6, 28, 120, 496, 2016, 8128, 32640, 130816, &c.; the aggregate of which series is the expression $\frac{1}{1-6+8}$; for this, when expanded, gives the series $1+6+28+120+496$, &c., in which all the perfect numbers likewise are contained.

31. If, therefore, to the series 6, 28, 496, 8128, 130816, 2096128, 33550336, 536854528, &c. viz.

if to the terms of the above series, omitting every other term after 28, and beginning from 6, the numbers 2, 4, 16, 64, 256, &c. are added, as below:

	2,	4,	16,	64,	256,	1024,	4096,	16384
	6,	28,	496,	8128,	130816,	2096128,	33550336,	536854528
1.	8,	32,	512,	8192,	131072,	2097152,	33554432,	536870912,

then the first sum will be equal to the third power of 2; the second to the fifth power of 2; the third to the ninth; the fourth to the thirteenth; the fifth to the seventeenth; the sixth to the twenty-first; the seventh to the twenty-fifth; the eighth to the twenty-ninth; and so on, there being always an interval after the first and second sums of four powers.

32. But the expression $\frac{1-8-96}{1-16}$, when evolved, gives the series $1+8+32+512+8192+131072$, &c., *ad infinitum;* and the expression $\frac{1-2-4}{1-4}$, gives the series $1+2+4+16+64+256$, &c. *ad infinitum.* But this latter, subtracted from the former, viz. $\frac{1-8-96}{1-16} - \frac{1-2-4}{1-4}$, gives $\frac{6-92+320}{1-20+64}$, which, when evolved, is the series $6+28+496+8128+130816$, &c.

33. The terms which, being multiplied by

numbers in a duple ratio, produce the series 1, 6, 28, 496, 8128, &c., are 1, 3, 7, 31, 127, 511, 2047, 8191, &c.; and the series equal to the last term of these numbers is $1+2+4+24+96+384$, &c. For $1=1$, $1+2=3$, $1+2+4=7$, $1+2+4+24=31$; and so of the rest. In which series it is remarkable that each term, after the first three terms, viz. after 1, 2, 4, is the $\frac{1}{4}$ of the following term. Thus 24 is the $\frac{1}{4}$ of 96; 96 is the $\frac{1}{4}$ of 384; and 384 is the $\frac{1}{4}$ of 1536; and so on.

34. After the first three terms, likewise, if each term is multiplied by 4, and 3 is added to the product, the sum will be the following term. Thus $31 \times 4 = 124$, and $124+3 = 127$. Thus, too, $127 \times 4 = 508$, and $508+3 = 511$; and again, $511 \times 4 = 2044$, and $2044+3 = 2047$; and so of the rest.

35. It is also remarkable that the sum of the divisors, consisting of 2 and its powers, is always equal, with the addition of unity, to the last quotient of the division. Thus $2+1=3$, the quotient in the division of 6. Thus, too, $2+4+1=7$, the last quotient in the division of 28; and $2+4+8$

$+16+1=31$, the quotient of 496; and so of the rest.

36. It is likewise possible to find an expression which, when evolved, will give the series $1+3+7+31+127$, &c. For this expression will be $\frac{1-2-4+8}{1-5+4}$.

37. As perfect numbers are resolved into their component parts, through a division by 2 and its powers, so that the sums arising from these as divisors, and from the quotients, together with unity, are respectively equal to the perfect numbers themselves; thus, also, the sum arising from a similar division of any term in the series $6+28+496+8128+130816+2096128$, &c., that is not a perfect number, will be equal to that term, though such division will not resolve it into all its parts.

38. But that the reader may more clearly apprehend my meaning, and be fully convinced of the truth of this assertion, the following instances are subjoined of a distribution of ten terms of this series into their parts by these divisors; among which terms four are perfect numbers.

(1)

2)6	3
3	2
	1
	6

(2)

2)28	14
2)14	7
7	4
	2
	1
	28

(3)

2)496	465
2)248	2
2)124	4
2)62	8
31	16
465	1
	496

(4)

2)8128	8001
2)4064	2
2)2032	4
2)1016	8
2)508	16
2)254	32
127	64
8001	1
	8128

(5)

2)130816	130305
2)65408	2
2)32704	4
2)16352	8
2)8176	16
2)4088	32
2)2044	64
2)1022	128
511	256
130305	1
	130816

(6)

2)2096128	2094081
2)1048064	2
2)524032	4
2)262016	8
2)131008	16
2)65504	32
2)32752	64
2)16376	128
2)8188	256
2)4094	512
2047	1024
2094081	1
	2096128

(7)

2)33550336	33542145
2)16775168	2
2)8387584	4
2)4193792	8
2)2096896	16
2)1048448	32
2)524224	64
2)262112	128
2)131056	256
2)65528	512
2)32764	1024
2)16382	2048
8191	4096
33542145	1
	33550336

(8)

2)536854528	536821761
2)268427264	2
2)134213632	4
2)67106816	8
2)33553408	16
2)16776704	32
2)8388352	64
2)4194176	128
2)2097088	256
2)1048544	512
2)524272	1024
2)262136	2048
2)131068	4096
2)65534	8192
32767	16384
536821761	1
	536854528

(9)		(10)	
2)8589869056	8589737985	2)137438691328	137438167041
2)4294934528	2	2)68719345664	2
2)2147467264	4	2)34359672832	4
2)1073733632	8	2)17179836416	8
2)536866816	16	2)8589918208	16
2)268433408	32	2)4294959104	32
2)134216704	64	2)2147479552	64
2)67108352	128	2)1073739776	128
2)33554176	256	2)536869888	256
2)16777088	512	2)268434944	512
2)8388544	1024	2)134217472	1024
2)4194272	2048	2)67108736	2048
2)2097136	4096	2)33554368	4096
2)1048568	8192	2)16777184	8192
2)524284	16384	2)8388592	16384
2)262142	32768	2)4194296	32768
131071	65536	2)2097148	65536
8589737985	1	2)1048574	131072
	8589869056	524287	262144
		137438167041	1
			137438691328

Here it is evident, in the first instance, that 3, 2, and 1, are all the possible parts of 6. For 3 is the half, 2 the third, and 1 the sixth part of 6; and it is likewise manifest that 14, 7, 4, 2, and 1, are all the parts of 28. Thus, too, in the division of 496 into its parts, 465 is the sum of the parts that are the quotients arising from the division by 2, and its powers. And as the half of 496 is 248, it is the same thing to divide 248 by 2, as to divide 496 by 4. For the same reason, it is the same thing to divide 124, the half of 248, by

2, as to divide 496 by 8; and to divide 62, the half of 124, by 2, as to divide 496 by 16. Hence the divisors of 496 are 2, 4, 8, 16, and the sum of these, added to 1 and to 465, is 496. In a similar manner, in the fourth instance, 8001 is the aggregate of the parts that are the quotients arising from the division by 2, and its powers. And the divisors of 8128 are 2, 4, 8, 16, 32, 64; the sum of which, added to 1, and to 8001, is equal to 8128; and so in the other instances.

39. Only eight perfect numbers have as yet been found, owing to the difficulty of ascertaining, in very great terms, whether a number is a prime or not: and these eight are as follow: 6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128. By an evolution of the expression $\frac{6-92+320}{1-20+64}$, to twenty terms, the reader will see at what distance these perfect numbers are from each other. But these twenty terms are as follow; those that are perfect numbers being designated by an asterisk:

*6, *28, *496, *8128, 130816, 2096128, *33550336, 536854528,

*8589869056, *137438691328, 2199022206976,

35184367894528, 562949936644096, 9007199187632128, 144115187807420416, 2305843008139952128, 36893488143124135936, 590295810341525782528, 9444732965670570950656, 151115727451553768931328.

Hence it appears that the eighth perfect number is the sixteenth term of the series produced by the expansion of $\frac{6-92+320}{1-20+64}$.

As, however, there are only eight perfect numbers in twenty terms of this series, it is evident that Ruffus, in his Commentary on the Arithmetic of Boetius, was greatly mistaken in asserting that every other term in the series is a perfect number.

40. It is remarkable in this series, that the terms alternately end in 6 and 8. This is also true of the four first perfect numbers; but the other four end indeed in 6 and 8, yet not alternately.

41. The other terms also of this series, from their correspondence with perfect numbers, may be called partially perfect. For both are resolved by 2, and its powers, into parts, the aggregates of which are equal to the wholes; and both are terminated by 6 and 8.

42. Again, in the series $1+8+32+512+8192$, &c., produced by the expansion of $\frac{1-8-96}{1-16}$, the half of each of the terms, after the first, is a square number. Thus the half of 8, of 32, of 512, &c., 4, 16, 216, are square numbers. And if each of the terms of the series $6+28+496+8128+130816$, &c. is doubled, the sum of the parts of each is a square number. Thus the sum of the parts of 12, the double of 6, is 16; the sum of the parts of 56, the double of 28, is 64; the sum of the parts of 992, the double of 496, is 1024; the root of which is 32: of the parts of 16256, the double of 8128, is 16384; the square root of which is 128: and of the parts of 261632, the double of 130816, is 262144; the square root of which is 512: and so of the rest; all the roots, after the second, increasing in a quadruple ratio. As this property also extends to the terms that are not perfect numbers, as well as to those that are, it shows, in a still greater degree, the correspondence of what I call the partially perfect with the completely perfect numbers.

43. Farther still, if 2 be subtracted from the number of the rank which any number, after 28, holds in the series 6, 28, 496, 8128, 130816,

&c., and the remainder be added to the number of the said rank, the sum will be the index of that power of 2, which, by multiplication with its corresponding number in the series, 3, 7, 31, 127, 511, &c., produced the perfect, or partially perfect number. Thus, if from 3 we subtract 2, and add 3 to the remainder 1, the sum 4 will indicate that the fourth power of 2, viz. 16, multiplied by the third term 31, will give the third term, viz. 496, in the series 6, 28, 496, 8128, &c. Thus also, if 2 be subtracted from 4, and 4 be added to the remainder, the sum 6 will indicate that the sixth power of 2, viz. 64, multiplied by the fourth term 127, will produce the fourth term 8128, of the series 6, 28, 496, 8128, &c.; and so of the rest.

44. The rule for obtaining a number which is either perfect, or partially perfect, in the series 6, 28, 496, 8128, 130816, &c., any term in this series being given, in the most expeditious manner, is the following. Multiply the given perfect or partially perfect number by 16, and add to the product twelve times the number in the duple series, from the multiplication of which with a corresponding number in the series 3, 7,

31, 127, 511, &c., the perfect, or partially perfect, number is produced, and the sum will be the next number in order in the series 6, 28, 496, &c. Thus $28 \times 16 = 448$; and 448, added to 12 times 4, *i. e.* to 48, is equal to 496. Thus, too, 496 $\times 16 = 7936$, and $7936 + 12 \times 16 \; (=192) = 8128$; and so of the rest.

45. Lastly, all the perfect numbers are found in the series of hexagonal numbers; which numbers are 1, 6, 15, 28, 45, 66, &c.; and the expression which is the aggregate of them, and when evolved, gives all of them in order *ad infinitum*, is $\frac{1+3}{1-3+3-1}$.

46. As Ozanam, in his Mathematical Recreations, does not mention what numbers are to be employed, in order to find other amicable numbers after 9363584, and 9437056 *, after

* "Schooten gives the following practical rule from Descartes, for finding amicable numbers, viz.: Assume the number 2, or some power of the number 2, such that if unity be subtracted from each of these three following quantities, viz. from three times the assumed number, also from six times the assumed number, and from eighteen times the square of the assumed number, the three remainders may

various trials to accomplish this, I found that if any number, by which two amicable numbers are obtained, is multiplied by 8, the product will be a number by which either two numbers may be obtained, the aggregate of all the parts of one of which is equal to the other; or two numbers, the aggregate of the parts of one of which only arising from a division by 2, and its powers, will be equal to the other. Thus, if 64 is multiplied by 8, the product will be 512. The three numbers, formed in the same way as the primes that produce amicable numbers, will be 1535, 3071, and 4718591. And two numbers, 4827120640, 4831837184, will be produced; the sum of the parts of the latter of which, arising from a division

be all prime numbers; then the last prime number being multiplied by double the assumed number, the product will be one of the amicable numbers sought, and the sum of its aliquot parts will be the other." — HUTTON'S *Mathem. Dict.*

This also is Ozanam's method of finding amicable numbers, which we have already given, and illustrated by examples. Mr. John Gough shows, that if a pair of amicable numbers be divided by their greatest common measure, and the prime divisors of these quotients be severally increased by unity, the products of the two sets, thus augmented, will be equal.

by 2 and its powers, with the addition of unity, will be equal to the former.

47. Again, if 512 is multiplied by 8, the product will be 4096. The three numbers corresponding to primes will be 12287, 24575, and 301989887: and the two imperfectly amicable numbers will be 2473599180800 and 2473901154304; the sum of the parts of the latter of which, arising from a division by 2 and its powers, with the addition of unity, will be equal to the former.

48. Farther still, the product of 4096, multiplied by 8, will be 32768. The three numbers, which are either primes or corresponding to primes, will be 98303, 196607, and 19327352831; and the two imperfectly or perfectly amicable numbers will be 1266618067910656 and 1266637395132416.

49. And, in the last place, if 32768 is multiplied by 8, the product will be 262144. The three numbers, which are either primes or corresponding to primes, will be 786431, 1572863, and 1236950581247; and the two imperfectly or perfectly amicable numbers will be 64851710939129446 and 64851834634082713 6.

50. In order that the reader may become acquainted with the method of obtaining the parts of perfectly amicable numbers, I shall give an instance of it in the two numbers 9363584 and 9437056. Let the first of these numbers then be divided by 2, and the powers of 2, viz. by 4, 8, 16, 32, 64, &c., till the division is stopped by a remainder, which is a prime number. These quotients, with the indivisible remainder, will be as below, 4681792, 2340896, 1170448, 585224, 292612, 146306, 73153. In the next place, as the two prime numbers, 191 and 383, are multiplied together, in order to produce the number 9363584, it is evident that these also are parts of it, and consequently they must be employed as the divisors of it. The quotient, therefore, of 9363584, divided by 191, is 49024; and the quotient of the same number, divided by 383, is 24448. Each of these quotients also may be divided by 2 and its powers. The quotients, therefore, arising from the division of 49024 by 2 and its powers, are 24512, 12256, 6128, 3064, 1532, 766; and the remainder is 383. But the quotients arising from a similar division of 24448, are 12224, 6112, 3056, 1528, 764, 382;

and the remainder is 191. The sum, therefore, of all these quotients, will be as below:

4681792
2340896
1170448
585224
292612
146306
73153
49024
24448
24512
12256
6128
3064
1532
766
383
12224
6112
3056
1528
764
382
191
9436801

And if to this sum the sum of 2 and its powers are added, together with unity, viz. if $1+2+4+8+16+32+64+128=255$ be added, the aggregate will be 9437056, the second of the amicable numbers.

In like manner, if 9437056 be divided by 2 and its powers, the quotients, and their aggregate, will be as follows:

4718528
2359264
1179632
589816
294908
147454
73727
9363329

And if to this sum 255 be added, as in the former instance, the aggregate will be 9363584, the first of the amicable numbers.

51. The first thing remarkable in these perfectly amicable numbers is, that the number 2 and its powers are employed in the production of all of them, and that they cannot be produced by any other number and the powers of it. For, as these amicable numbers are images of true friendship, this most clearly shows that such friendship can only exist between two persons.

52. In the next place, the numbers 3, 6, and 18, which are used in the formation of these numbers, perspicuously indicate perfection, and are therefore images of the perfection of true friendship. For 3 and 6 are the first perfect numbers; the former, from being the paradigm of *the all*, comprehending in itself beginning, middle, and end, and the latter from being equal

to all its parts; and 18 is produced by the multiplication of 6 by 3.

53. In the third place, the paucity of these numbers most beautifully adumbrates the rarity of true friendship. For between 1 and 1000 there are only two. In like manner, between 284 and 20000 there are only two; and between 18416 and ten millions there are only two.

54. In the fourth place, the greater the numbers are in the series of perfectly and imperfectly amicable numbers, the nearer they approach to a perfect equality. Thus, for instance, the exponent of the ratio of 220 to 284 is $1\frac{64}{220}$, and by reduction $1\frac{16}{55}$. But the exponent of the ratio of 17296 to 18416 is $1\frac{1120}{17296}$; and by reduction, $1\frac{70}{1081}$. And $\frac{70}{1081}$ is much less than $\frac{16}{55}$. Again, the exponent of the ratio of 9363584 to 9437056 is $1\frac{73472}{9363584}$; and by reduction, $1\frac{287}{73153}$. And $\frac{287}{73153}$ is much less than $\frac{70}{1081}$. In a similar manner it will be found that the exponents of the ratios of the succeeding amicable numbers will continually decrease; and, consequently, that the greater two amicable numbers become, the nearer they approach to an equality with each other. Indeed, in the amicable numbers

after the first three, this is obvious by merely inspecting the numbers themselves. For, in the amicable numbers 4827120640 and 4831837184, the first two numbers, from the left hand to the right, are the same in each, viz. 4 and 8. In the two amicable numbers, 2473599180800 and 2473901154304, the first four numbers, 2473, are the same in each. In the two amicable numbers, 12666180679910656 and 12666373951832416, the first five numbers are the same in each. This is also the case with the two amicable numbers which immediately succeed these, viz. 6485171093912994464 and 6485183463408327136. And in the two amicable numbers which are next but one to these last, viz. in the numbers 170005188316757680455680 and 170005193383307194138624, the first seven numbers, from the left hand to the right, are the same in each; by all which it appears, that the greater two amicable numbers are, the more figures in the one are the same as those in the other, and consequently that their approximation to a perfect equality is greater.

55. The following are the remarkable properties of the prime numbers, and the numbers

corresponding to primes, from which bo...
of amicable numbers are produced.

These numbers are as follow:

5 .	11 .	71
23 .	47 .	1151
191 .	383 .	73727
1535 .	3071 .	4718591
12287 .	24575 .	301989887
98303 .	196607 .	19327352831.

In the first place, 2×5 and $+1 = 11$, 2×23 and $+1 = 47$, 2×191 and $+1 = 383$, 2×1535 and $+1 = 3071$, 2×12287 and $+1 = 24575$, and 2×98303 and $+1 = 196607$. And thus the double of the first number in each rank, added to unity, is equal to the second number in the same rank.

In the next place, the first number of the first rank, multiplied by 4 and added to 3, will be equal to the first number of the second rank. Thus also the second number 11×4, and added to $3 = 47$, the second number of the second rank. But 23×8 and $+7 = 191$, 47×8 and $+7 = 383$, 191×8 and $+7 = 1535$, 383×8 and $+7 = 3071$, 1535×8 and $+7 = 12287$, 3071×8 and $+7 = 24575$, 12287×8 and $+7 = 98303$, and 24575×8 and $+7 = 196607$.

Again, 47×4 and $+3=191$, 383×4 and $+3=1535$, 3071×4 and $+3=12287$, and 24575×4 and $+3=98303$.

Again, $\frac{1151}{71}=16$, with a remainder 15. $\frac{73727}{1151}=64$, with a remainder 63. $\frac{4718591}{73727}=64$, and the remainder is 63. And $\frac{301989887}{4718591}=64$, with the same remainder 63. And so of the rest *ad infin.*, the quotient and remainder being always 64 and 63.

56. Hence infinite series of all these may easily be obtained, viz. of the two first terms in each rank, the first rank excepted; and of the third term in each rank, the two first ranks excepted. For $\frac{23+7+7+7}{1-8}$, &c. *ad infin.* $=23+191+1535+12287$, &c., $\frac{47+7+7+7}{1-8}$, &c. *ad infinitum*, $=47+383+3071+24575$, &c. And $\frac{73727+63+63+63}{1-64}$, &c. $=73727+4718591$, &c. But these expressions, when reduced, will be $\frac{23-16}{1-9+8}$, $\frac{47-40}{1-9+8}$, and $\frac{73727-73664}{1-65+64}$.

57. Another remarkable property of these numbers is this, that the product of the two first in each rank, subtracted from the third number

in the same rank, leaves a remainder equal to the aggregate of those two first. Thus $5 \times 11 = 55$, and $71 - 55 = 16$. But $16 = 5 + 11$. Thus, too, $23 \times 47 = 1081$, and $1151 - 1081 = 70$, $= 23 + 47$. And thus $191 \times 383 = 73153$, and $73727 - 73153 = 574 = 191 + 383$. And so of the rest. Hence the third, less by the sum of the two other numbers, will be equal to the product of the two first. From the expressions, therefore, before given, and from what is now shown, it will be easy to find an expression which, when evolved, will give an infinite series of the products of the two first numbers in each rank. And this expression (the first term of it being the product of the two first numbers in the third rank, in order that it may be in the same rank with the expression $\frac{73727-73664}{1-65+64}$), will be $\frac{73153-699337+1179656-503472}{1-74+657-1096+512}$.

58. All these amicable numbers, therefore, in order after 17296, 18416, may be found, if the first term of the infinite series, arising from the expansion of each of these expressions, is multiplied by 128; the second term of each by 1024;

the third term by 8192; and so on, the multipliers always increasing in an octuple ratio.

59. That the reader may see the truth of what we have asserted concerning this species of imperfectly amicable numbers exemplified, the following instances are added of the resolution of two of them into parts by 2 and its powers.

In the first place, the sum of the parts of the number 4831837184 is equal to 4827120640, as is evident from the following division:

	Divisors by 2, and its powers, with the addition of unity.
2)4831837184	
2)2415918592	1
2)1207959296	2
2)603979648	4
2)301989824	8
2)150994912	16
2)75497456	32
2)37748728	64
2)18874364	128
2)9437182	256
4718591	512
4827118593	1024
+2047	2047 Total.
4827120640	

60. Of every square number, consisting of more than one digit or figure, the ultimate sum of the digits will always be either 1, or 4, or 7, or 9, or 10. Thus in 16, the square of 4, the sum

of the digits is $1+6=7$; in 25, the square of 5, the sum of the digits is 7; in 36 the sum is 9; in 49 the sum is 13, and ultimately $1+3=4$; in 64 the sum is 10; in 81 it is 9; and in 100 it is 1. Again, in 121, the square of 11, the sum of the digits is 4; in 144 it is 9; in 169 it is 16, and ultimately 7; in 196 it is also 16, and ultimately 7; in 225 it is 9; in 256 it is 13, and ultimately 4; in 289 it is 19, and ultimately 10; in 324 it is 9; in 361 it is 10; and in 400 it is 4. And so in all other instances *ad infinitum.*

But of every cube number, consisting of more than one digit, the ultimate sum of the digits will always be either 1, or 8, or 9, or 10. Thus, in the cubes of the numbers 3, 4, 5, 6, 7, 8, 9, 10, viz. 27, 64, 125, 216, 343, 512, 729, 1000, the sum of the digits of 27 is 9; of 64 it is 10; of 125 it is 8; of 216 the sum is 9; of 343 it is 10; of 512 it is 8; of 729 it is 18, and ultimately 9; and of 1000 it is 1. Thus, too, in the cubes of the numbers 11, 13, 14, 15, 16, 17, 18, 19, 20, viz. 1331, 1728, 2197, 2744, 3375, 4096, 4913, 5832, 6859, 8000, the sum of the digits of 1331 is 8; of 1728 is 18, and ultimately 9;

of 2197 is 19, and ultimately 10; of 2744 is 17, and ultimately 8; of 3375 is ultimately 9; of 4096 is also ultimately 9; of 4913 is ultimately 8; of 5832 is ultimately 9; of 6859 is ultimately 10; and of 8000 is 8. And the like will take place in all other cube numbers.

61. Anatolius, as we are informed by the author of Theologumena Arithmeticæ *, says, "that the tetrad is called *justice*, because the square produced from it, *i. e.* the το εμβαδον, or space contained within the sides of a square, each of whose sides is 4, is equal to the perimeter. For $4 \times 4 = 4 + 4 + 4 + 4$. But of the numbers prior to 4, the perimeter is greater than the εμβαδον; and of the numbers posterior to 4, the perimeter is less than the εμβαδον †." Thus the square of 2 is 4, but the perimeter is 8; and the square of 3 is 9, but the perimeter is 12. And again,

* See the edition of this work by Frid. Astius, Lipsiæ, 1817, 8vo.

† Καλειται δε αυτη, ως φησιν ο Ανατολιος, δικαιοσυνη, επει το τετραγωνον το απ' αυτης, τουτεστι το εμβαδον τη περιμετρω ισον· των μεν γαρ προ αυτης η παραμετρος (lege περιμετρος) του εμβαδου του τετραγωνου μειζων, των δε μετ' αυτην η περιμετρος του εμβαδου ελαττων, επ' αυτης δε ιση.

the square of 5 is 25, but the perimeter is 20; the square of 6 is 36, but the perimeter is 24; the square of 7 is 49, but the perimeter is 28; the square of 8 is 64, but the perimeter is 32; the square of 9 is 81, but the perimeter is 36; and the square of 10 is 100, but the perimeter is 40. Hence it appears, that of the numbers prior to 4, the perimeters exceed the squares; but of the numbers posterior to 4, the perimeters are less than the squares by the following differences. The square of 5, *i. e.* 25, exceeds the perimeter 20 by 5, the root of the square; 36 exceeds 24 by 12, the double of the root 6; 49 exceeds 28 by 21, the triple of the root 7; 64 exceeds 32 by 32, the quadruple of the root 8; 81 exceeds 36 by 45, the quintuple of the root 9; and 100 exceeds 40 by 60, the sextuple of the root 10: and so of the rest. The perimeters also exceed each other by 4.

In cubes, the cube of 6, *i. e.* 216, is analogous to the square of 4. For, as the latter is equal to its perimeter, so likewise is the former; for each of its sides is a square equal to 36; and as there are six sides, the whole perimeter is 216.

Again, $\frac{1+2+3}{1-2+1}$ is, when expanded, $1+4+10+16+22+28$, &c., and is a series containing in itself all the perfect numbers after 6. But the numerator of this expression is 6, in a distributed form.

The expression $\frac{1+6-3}{1-2+1}$ is, when expanded, the series $1+8+12+16+20$, &c., and is equal to the aggregate of the perimeters of all the squares. But the aggregate of all the squares is $\frac{1+1}{1-3+3-1}$*, the last term of which is $\frac{1+1}{1-2+1}$†. Hence the aggregate of the perimeters of all the squares is double of the last square in the infinite series of squares.

Thus, too, the expression $\frac{1+21-15+5}{1-3+3-1}$ is, when expanded, the series $1+24+54+96+150$, &c., and is equal to the aggregate, in a distributed form, of all the perimeters, or bounding superficies, of the infinite series of cubes, $1+8+27$

* For this expression, when expanded, is $1+4+9+16+25+36$, &c.

† For this, when expanded, is the series $1+3+5+7+9+11$, &c.; the aggregate of which is evidently equal to the last term of the series $1+4+9+16+25$, &c.

$+64+125$, &c. $= \frac{1+4+1}{1-4+6-4+1}$. But the last term of this series, or the last cube, is $\frac{1+4+1}{1-3+3-1}$ *. And therefore the distributed aggregate of all the perimeters, *i. e.* $\frac{1+21-15+5}{1-3+3-1}$, is the double of the last cube, *i. e.* of $\frac{1+4+1}{1-3+3-1}$. For the numerator $1+21-15+5$ is equal to 12, and the numerator $1+4+1$ is equal to 6.

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* For this expression, when expanded, is the series $1+7+19+37+61$, &c. And 1 is the first cube, $1+7=8$ is the second cube, $1+7+19=27$ is the third cube, $1+7+19+37=64$ is the fourth cube, and $1+7+19+37+61=125$ is the fifth cube; and so of the rest. And, therefore, the aggregate of this series is equal to the last cube.

THE END.

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